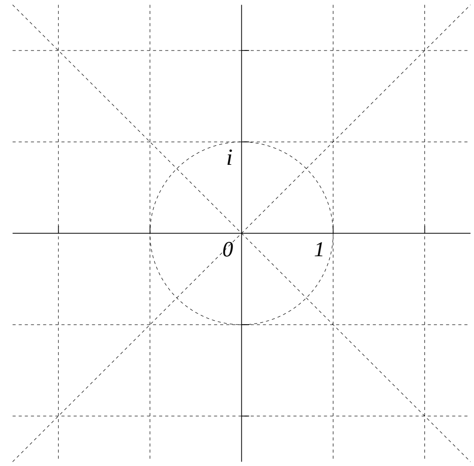
(20) 1. In this problem U will be the region in the complex plane defined by the inequalities $r = |z| > 1$ <u>and</u> $-\frac{\pi}{2} < \operatorname{Arg} z < \frac{3\pi}{4}$.

a) Sketch the region U on the axes given.



Is U a connected open set? (Yes | No)

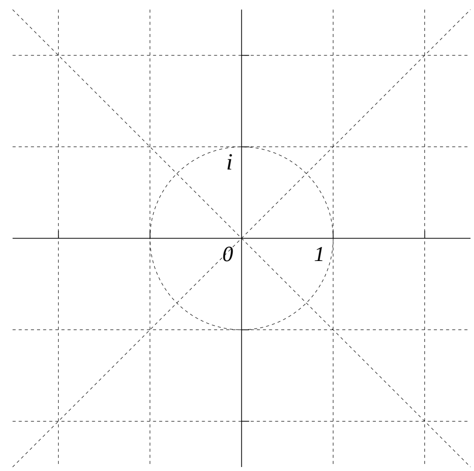
ANSWER: ____________

Is U a simply connected open set? (Yes | No)

ANSWER: ____________

b) Suppose $F(z) = z^2$. If $z = re^{i\theta}$, write a formula for $F(z)$ in complex exponential form.

ANSWER: ____________



If $V = F(U)$, the image of U under F, the collection of all values of F with domain restricted to U, sketch the region V on the axes given.

Is V a connected open set? (Yes | No)

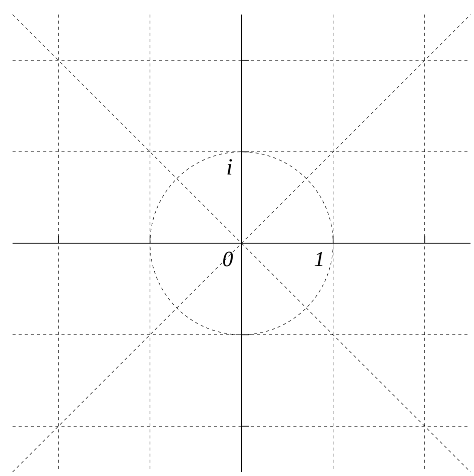
ANSWER: ____________

Is V a simply connected open set? (Yes | No)

ANSWER: ____________

c) Suppose $G(z) = \frac{1}{z}$. If $z = re^{i\theta}$, write a formula for $G(z)$ in complex exponential form.

ANSWER: ____________



If $W = G(U)$, the image of U under G, the collection of all values of G with domain restricted to U. sketch the region W on the axes given.

Is W a connected open set? (Yes | No)

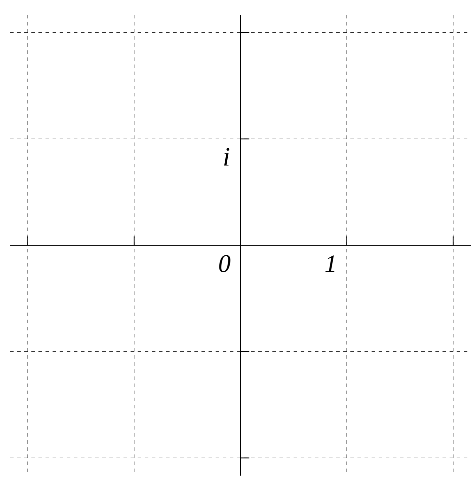
ANSWER: ____________

Is W a simply connected open set? (Yes | No)

ANSWER: ____________

(10) 2. Find $\operatorname{Arg} z$ if $z = (1+i)^i$.

(12) 3. Describe all solutions of $z^3 = 2i$ algebraically in either rectangular or polar form. Sketch these solutions on the axes provided.

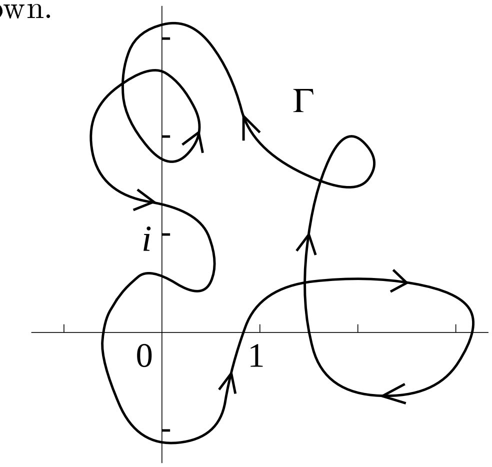


(18) 4. a) Suppose $c(x, y)$ is a harmonic function, so $\dfrac{\partial^2 c}{\partial x^2} + \dfrac{\partial^2 c}{\partial y^2} = 0$.

Prove that $f = \dfrac{\partial c}{\partial x} - i\dfrac{\partial c}{\partial y}$ is complex analytic.

b) Verify that $w(x, y) = \cos x \cosh y$ is harmonic, and find one harmonic conjugate.

(25) 5. Compute $\displaystyle\int_\Gamma \frac{z^2}{e^{2\pi i z}-1}\,dz$ where Γ is the closed curve shown.



(20) 6. a) If k is a real number with $-1 < k < 1$, derive the Laurent series representation

$$\frac{k}{z-k} = \sum_{n=1}^{\infty} \frac{k^n}{z^n} \qquad (|k| < |z| < \infty)\,.$$

b) Write $z = e^{i\theta}$ in the equation obtained in part a) and then equate real parts on each side of the result to derive the summation formula

$$\sum_{n=1}^{\infty} k^n \cos n\theta = \frac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}\,.$$

(25) 7. If A is real and $A > 1$, `Maple` reports that

$$\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+A^2)} = \left(\frac{1}{2}\right)\left(\frac{\pi}{A+1}\right).$$

Check this assertion using the method of residues. Show clearly the contour of integration and any residue computation. Explain why some integral tends to 0 in the limit.

(16) 8. What is the radius of convergence of the Taylor series expansion of

$$h(z) = \frac{e^z}{(z-1)(z+2)}$$

when expanded around $z = i$? Give a numerical answer. Justify why the series must converge with at least that radius <u>and</u> why it can't have a larger radius.

Note Actual computation of the series is not practical.

(20) 9. Find the order of the pole of

$$H(z) = \frac{1}{(6\sin z + z^3 - 6z)^2}$$

at $z = 0$.

(20) 10. Suppose T is the inside of the square with corners $1+i$, $-1+i$, $-1-i$, and $1-i$, and S is the inside of the square with corners $3+3i$, $-3+3i$, $-3-3i$, and $3-3i$. Suppose also that $f(z)$ is any entire function. Let M be the maximum of $|f''(z)|$ on T and let N be the maximum of $|f(z)|$ on S. Show that

$$M \leq \frac{1}{2}N\,.$$

Hint Begin by writing some complex variables formula connecting f'' and f.

(14) 11. Suppose $F(z) = z^3\left(\frac{1}{z+1} + \frac{1}{(z-1)^3}\right)$, and C is a simple closed curve which does *not* pass through 1 or -1. What are all possible values of $\int_C F(z)\,dz$ (and why)? Sketch examples of C's which will give each value you list.

Final Exam for Math 403, section 5

May 7, 2001

NAME ____

Do all problems, in any order.
Show your work. An answer alone will not receive full credit.
Answers must be supported by computation and explanation.
No notes or calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	20	
2	10	
3	12	
4	18	
5	25	
6	20	
7	25	
8	16	
9	20	
10	20	
11	14	
Total Points Earned:		