April 8, 2001

Review Problems for the second exam in section 5 of Math 403

This exam will test material covered since the first exam: basically what we've done from chapters 4 and 5 of the text. In the problems below, the letter $\mathcal{C}$ will be used to represent the boundary of "the unit circle": the circle of radius 1 centered at 0, oriented counterclockwise as usual.

1. Compute $\displaystyle\int_{\mathcal{C}} \frac{\sin z}{z^4}\, dz$ in several ways. For example, you could use the CIF for derivatives, or you could use the power series expansion for $\sin z$ and integrate directly.

2. Compute $\displaystyle\int_{\mathcal{C}} \frac{dz}{9+z^2}$ in several ways. For example, you could use Cauchy's Theorem, or you could find the Taylor series expansion of the integrand (via a geometric series argument) and integrate directly.

3. Discuss whether the following statement is always correct:

> The integral of an analytic function around a simple closed curve is always 0.

If the statement is correct, explain why it is correct. If it is not, provide some simple examples to show that it can be false. Also, give some additional hypothesis which will make the conclusion of the statement valid.

4. Evaluate the integral of $\displaystyle\frac{1}{(1-z)^3}$ over the following curves:

a) The circle of radius $\frac{1}{2}$ centered at 0.
b) The circle of radius $\frac{1}{2}$ centered at -1.
c) The circle of radius $\frac{1}{2}$ centered at 1.

5. Suppose that f is entire and you know that there is a real constant $A > 0$ so that $|f(z)| \le A|e^z|$ for all z. Show that f must then be some constant multiple of e^z.
Hint The exponential function is never ... and things which aren't ... have multiplicative inverses – divide by them, and finish the problem with almost no effort.

6. Liouville's Theorem states that entire functions with "no growth" as $z \to \infty$ must be constant. In fact, functions with even "slow growth" must be constant. Please verify a version of this:

> If f is an entire function satisfying $|f(z)| \le A\sqrt{|z|}$ for some fixed $A > 0$ and for all z with $|z|$ large, then f must be constant.

This can be done with the Cauchy inequalities, similar to the verification of Liouville's Theorem: show that $f'(z_0)$ must always be 0.

7. Suppose that the series $\sum_{n=0}^{\infty} a_n(3+4i)^n$ converges (here the a_n's are complex numbers). What can you then deduce about the series $\sum_{n=0}^{\infty} a_n(-1+2i)^n$? Explain your conclusions.

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8. What is the radius of convergence of the Taylor series expansion of

$$f(z) = \frac{e^z}{(z-1)(z+1)(z-2)(z-3)}$$

when expanded around $z = i$? Give a numerical answer. Justify why the series must converge with at least that radius <u>and</u> why it can't have a larger radius. I don't think that actual computation of the series is practical!

9. Find two distinct Laurent series for the function

$$g(z) = \frac{1+3z}{4z^2+z^4}$$

and in each case specify the annulus in which the series is valid. Find explicit values of the coefficients of z^{-36} and z^{35} in each case.

10. a) Find a Laurent series for $e^{-\frac{1}{z}}$ and describe those z's for which it is valid.

b) Use your answer to a) to compute $\displaystyle\int_{\mathcal{C}} z^4 e^{-\frac{1}{z}}\,dz$.

11. a) If h is defined by

$$h(z) = \frac{\sin(z^2)}{2+z^3}$$

find the terms up to and including degree 5 in the Taylor series centered at 0 of h.

b) Use your answer to a) to compute $h^{(5)}(0)$.