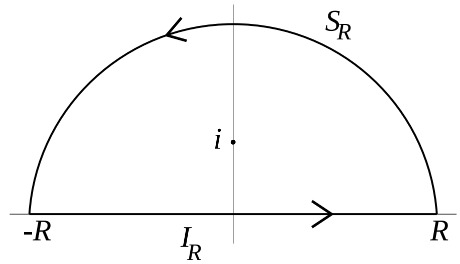
640:403:01 **Answers to the Second Exam** **4/28/2001**

(16) 1. Use the Residue Theorem to compute $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$.



Answer We apply the Residue Theorem to the function $f(z) = \frac{1}{(1+z^2)^2}$ on the simple closed curve $I_R + S_R$, where $I_R = [-R, R]$ is an interval on the real axis, and S_R is the upper semicircle: $|z| = R$ and $\mathrm{Im}\, z \geq 0$. The curve is oriented counterclockwise, and $R > 1$. $f(z)$ has isolated singularities at $\pm i$, and since $f(z) = \frac{1}{(z-i)^2(z+i)^2}$, the isolated singularities are poles of order 2. The singularity at i is inside the closed curve. If we write $f(z) = \frac{H(z)}{(z-i)^2}$, then the residue of $f(z)$ at $z = i$ is just $H'(i)$ since $H(z) = H(i) + H'(i)(z-i) + \textit{higher order terms}$. Here $H(z) = \frac{1}{(z+i)^2}$ so $H'(z) = \frac{-2}{(z+i)^3}$ and $H'(i) = \frac{1}{4i}$. As $R \to \infty$, $\int_{I_R} f(z)\, dz \to \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$. If $|z| = R$, $|1 + z^2| \overset{\nabla}{\geq} R^2 - 1$ so that for $R > 1$, $|\int_{S_R} f(z)\, dz| \leq \pi R \cdot \frac{1}{(R^2-1)^2}$ by the ML inequality. Therefore as $R \to \infty$, $\int_{S_R} f(z)\, dz \to 0$. When $R > 1$, the Residue Theorem shows that $\int_{I_R + S_R} f(z)\, dz = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$. As $R \to \infty$, we see that $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$.

(16) 2. Find $R > 0$ so that all roots of the polynomial $P(z) = z^5 + 12z^4 - (1+i)z^2 + 9z - 3$ are inside the circle $|z| = R$.

Answer Take $R = 100$, $f(z) = z^5$, and $g(z) = 12z^4 - (1+i)z^2 + 9z - 3$. For $|z| = 100$, $|f(z)| = 10^{10}$ while $|g(z)| \overset{\Delta}{\leq} 12|z|^4 + \sqrt{2}|z|^2 + 9|z| + 3 \leq 10^8(12 + \sqrt{2} + 9 + 3) < 10^{10}$. We use Rouché's Theorem: $f(z)$ and $f(z) + g(z) = P(z)$ must have the same number of zeros inside the circle $|z| = 100$. But $f(z)$ has a zero of multiplicity 5 at 0. So $P(z)$ must have five zeros inside $|z| = 100$ and $P(z)$, a polynomial of degree 5, can have at most five zeros.

(16) 3. Use the Residue Theorem to compute $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$.

Answer Since $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \int_0^{2\pi} \frac{2e^{i\theta}}{10e^{i\theta}+3(e^{i\theta})^2+3}\, d\theta$. If $z = e^{i\theta}$ so $dz = ie^{i\theta}\, d\theta$, the definite integral is recognized as a parameterization of a line integral around the unit circle, $|z| = 1$: $\frac{1}{i}\int_{|z|=1} \frac{2}{3z^2+10z+3}\, dz$. $3z^2 + 10z + 3 = 0$ when $z = \frac{-10\pm\sqrt{100-36}}{6} = \frac{-10\pm 8}{6}$. The integrand has isolated singularities which are both poles of order 1 at -3 and at $-\frac{1}{3}$. To use the Residue Theorem, we need the residue inside $|z| = 1$. Since $\frac{2}{3z^2+10z+3} = \frac{2}{3(z+\frac{1}{3})(z+3)}$ the residue of the integrand at $-\frac{1}{3}$ is $\frac{2}{3(z+3)}$ at $z = -\frac{1}{3}$, which is $\frac{1}{4}$. Then the Residue Theorem gives the value of the desired integral: $2\pi i \cdot \frac{1}{i} \cdot \frac{1}{4} = \frac{\pi}{2}$.

(14) 4. Suppose $Q(z) = \frac{(e^z-1)^2}{z^4}$. Identify as precisely as possible the type of the isolated singularity at 0 of $Q(z)$: is it removable, a pole, or essential? If it is a pole, find the order of the pole. Find the first two non-zero terms of the Laurent series of $Q(z)$ at 0. Find the residue of $Q(z)$ at 0.

Answer We know $e^z = 1 + z + \frac{z^2}{2} + \textit{higher order terms}$ so that $\frac{(e^z-1)^2}{z^4} = \frac{\left(z+\frac{z^2}{2}+h.o.t.\right)^2}{z^4} = \frac{z^2+z^3+h.o.t.}{z^4} = \frac{1}{z^2} + \frac{1}{z} + h.o.t.$ and we can read off the answers: $Q(z)$ has a pole of order 2 at 0, its Laurent series begins $\frac{1}{z^2} + \frac{1}{z}$, and its residue at 0 is 1.

(12) 5. Suppose $f(z)$ is an entire function (analytic in the whole plane) and that $|f(z)| \le |e^z|$ for all complex numbers z. Show that there must be a complex number C with $|C| \le 1$ so that $f(z) = Ce^z$ for all z.

Answer Since e^z is never 0, $F(z)) = \frac{f(z)}{e^z}$ is an entire function. The hypotheses say that $|F(z)| \le 1$ for all z. Liouville's Theorem implies that $F(z)$ is constant, and it must be a constant C with $|C| \le 1$. So $\frac{f(z)}{e^z} = C$ for all z, and $f(z) = Ce^z$ for all z, as desired.

(14) 6. a) Compute $\int_{|z|=1} \frac{e^z}{z}\,dz$ using any applicable theorem.

Answer We could use the Residue Theorem again, but the Cauchy Integral Formula for $z = 0$ also applies. The result is $2\pi i$ multiplied by the value of e^z at $z = 0$. This value is just 1, so the integral's value is $2\pi i$.

b) Use the answer given for part a) to find the exact values of $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta$ and of $\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta$.

Answer If $z = e^{i\theta}$, $e^z = e^{\left(e^{i\theta}\right)} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta}e^{i\sin\theta} = e^{\cos\theta}\big(\cos(\sin\theta) + i\sin(\sin\theta)\big)$. Also $dz = ie^{i\theta}\,d\theta$ so that $\frac{dz}{z} = \frac{ie^{i\theta}\,d\theta}{e^{i\theta}} = i\,d\theta$. Therefore $\int_{|z|=1} \frac{e^z}{z}\,dz = \int_0^{2\pi} e^{\cos\theta}\big(\cos(\sin\theta) + i\sin(\sin\theta)\big)i\,d\theta$. Part a) tells us this is $2\pi i$. We see that $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = 2\pi$ and $\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta = 0$ by separating the real and imaginary parts. This is consistent with `Maple`'s approximations.

Comment The versions of `Maple` I have at home can't compute this integral symbolically, but the latest version at Rutgers can.

(12) 7. The following information is known about a function, $F(z)$:

i) $F(z)$ is defined and analytic for all $z \ne 0$.
ii) $F(i) = 3$.
iii) For all positive integers, n, $F(\frac{1}{n}) = 0$.

a) What kind of isolated singularity must $F(z)$ have at 0? Explain your answer.

Answer If 0 were a pole then $|F(\frac{1}{n})| \to \infty$ as $n \to \infty$ and this is false by iii). If 0 were a removable singularity, then $F(0) = \lim_{n\to\infty} F(\frac{1}{n}) = 0$. If this is true, $F(z)$ with the value 0 at $z = 0$ would be entire. Then $F(z)$ would be 0 on a sequence with a limit point. Such a function would need to be 0 *everywhere* by the Identity Theorem ("Two functions analytic in a connected open set which agree on a set with a limit point must actually agree everywhere in the set."), but this would contradict ii), that $F(i) = 3$. The only alternative is that $F(z)$ has an essential singularity at $z = 0$.

b) What is the radius of convergence of the Taylor series expansion centered at $z = i$ of the function $F(z)$? Explain your answer.

Answer The radius of convergence is at least 1, because $F(z)$ is certainly analytic in a disc of radius 1 centered at i. If the radius of convergence were larger than 1, the sum would represent an analytic function agreeing with $F(z)$ in an open disc, and therefore agreeing with $F(z)$ for all $z \ne 0$ in the disc (using the Identity Theorem again). Then $F(z)$ would have a removable singularity at 0 because the Taylor series would behave like an analytic function near 0. Since $F(z)$ has an essential singularity at 0, this is impossible. So the radius of convergence must be exactly 1.