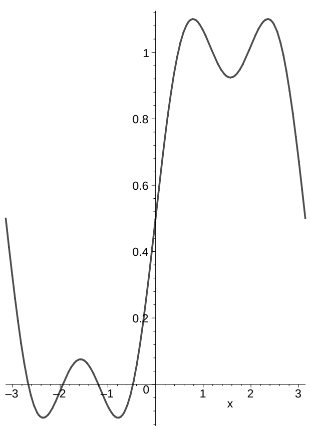
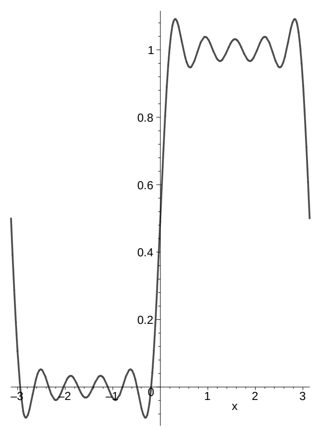
Math 403, section 1 **A Fourier series example** March 1, 2002

Just as power series are infinite linear combinations of monomials, $\{x^n\}_{n=0}^{\infty}$, Fourier series (the specific series given here is called a Fourier sine series) are infinite linear combinations of the functions $\{\sin nx\}_{n=0}^{\infty}$. Power series have very nice behavior inside their radius of convergence: "The sum is continuous inside the radius of convergence" and "The series can be differentiated 'term-by-term' inside the radius of convergence". Fourier series generally aren't as nice as power series, and this example is presented to explain why these properties of power series should *not* be accepted as CLEAR.

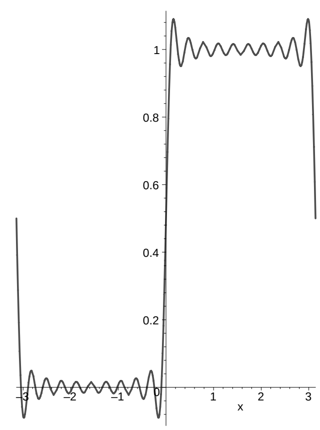
The example is the series $\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{2n-1}$. If $S_N(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{N}\frac{\sin((2n-1)x)}{2n-1}$ (the N^{th} partial sum), it can be proved that $S_N(x)\to H(x)$ for all x as $N\to\infty$, where H is the function defined piecewise by $H(x)=\begin{cases}0, & \text{if } -\pi<x<0\\ \frac{1}{2}, & \text{if } x=0\\ 1, & \text{if } 0<x<\pi\end{cases}$. The behavior of this example also certainly isn't CLEAR, but it appears in any course or text discussing Fourier series. The `Maple` graphs of some partial sums are shown here.



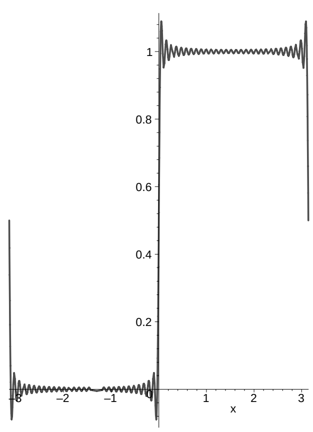
Graph of $S_2(x)$



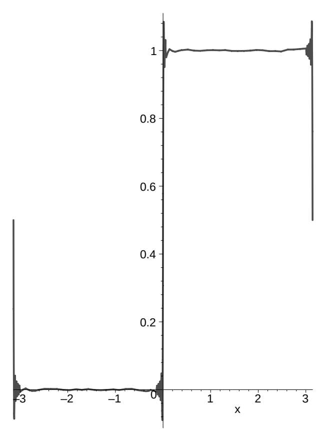
Graph of $S_5(x)$



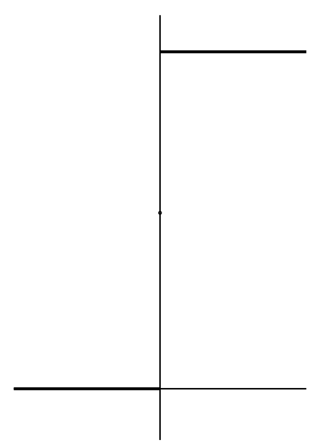
Graph of $S_{10}(x)$



Graph of $S_{30}(x)$



Graph of $S_{100}(x)$



Graph of $H(x)$