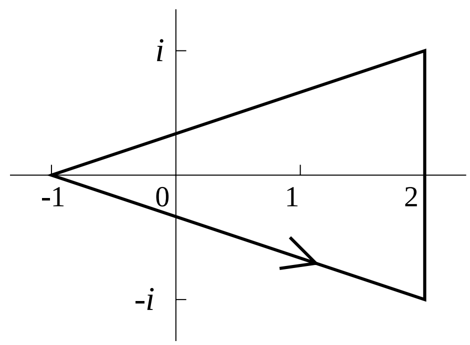
April 15, 2002

Review Problems for the second exam in section 1 of Math 403

The review material is in three parts. This part contains the questions. Another page (available through the course homepage on the web) has **bald answers** (or hints, where appropriate) to the questions. Much more **extended discussion** of the answers is also available on the web through the course homepage. Students who may want a hint or a numerical answer can look at the bald answer page, and those desiring a more extended discussion can look for that. I hope this arrangement is helpful.

1. Suppose $f(z)$ is analytic in the region $0 < |z| < 3$, and suppose that $f(\frac{1}{n}) = 0$ and $f(\frac{1}{n}i) = 1$ for all positive integers, n. Explain why there must be a sequence $\{z_n\}$ with $\lim_{n\to\infty} z_n = 0$ and so that $\lim_{n\to\infty} f(z_n)$ exists and is 2.

2. Suppose $g(z) = \frac{5}{z} + \frac{-3}{(z-1)^2}$, and that γ is the closed curve which is a triangle with vertices at -1, $2-i$, and $2+i$.



a) What is $\int_\gamma g(z)\,dz$?

b) What is $\int_\gamma g'(z)\,dz$?

c) What is $\int_\gamma \big(g(z)\big)^2\,dz$?

3. Suppose that $k(z)$ is an entire function satisfying the inequality $|k(z)| \le A\ln(|z|) + B$ for all $z \ne 0$. Prove that $k(z)$ is constant.

4. Suppose that $h(z)$ is an entire function, and that $|h(z)| \le |z^7|$ for all z. Prove that $h(z) = Cz^7$ where C is a complex number with $|C| \le 1$.

5. If $m(z) = \frac{1}{z - \sin z}$, what is the type of the singularity of $m(z)$ at 0? Find the first two non-zero terms of the Laurent series for $m(z)$ at 0. What is the residue of $m(z)$ at 0?

6. Suppose that $P(z) = z^5 + 12z^2 + 2$.

a) Explain why $P(z)$ has two roots (counted with multiplicity) inside the unit circle $|z| = 1$.

b) Explain why $P(z)$ has five roots (counted with multiplicity) inside the circle of radius 10 centered at 0.

c) How many roots does $P(z)$ have in the annulus $1 < |z| < 10$?

7. What is the radius of convergence of the Taylor series expansion centered at $z = i$ of the function $Q(z) = \frac{e^z}{(z-1)(z+1)(z-2)(z-3)}$? Give an exact answer. Justify why the series must converge with at least that radius and why it can't have a larger radius. (Actual computation of the series is not practical and is not requested.)

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8. Suppose $R(z)$ is the function defined by $R(z) = \dfrac{z+1}{z(z-1)}$. Find the Laurent series representing $R(z)$ in the annulus $0 < |z| < 1$. Be sure to explain *why* the series you write is valid in that annulus. Find explicit values of the coefficients of z^{10} and z^{-10} in the series. Hint: Use partial fractions.

9. The function $S(z) = (1+3z)e^{(z^2)}$ is analytic near 0.

a) Use results about power series of familiar functions to find terms up to and including degree 4 in the Taylor series of S centered at $z = 0$.

b) Use your answer to a) to compute $S^{(4)}(0)$.

10. Use the Residue Theorem to compute $\displaystyle\int_0^{2\pi} \frac{2}{3+\sin\theta}\, d\theta$.

11. Use the Residue Theorem to compute $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(4+x^2)}$.

12. Use the Residue Theorem to compute $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\, dx$ if a is a positive real number.

13. Can there be a function $q(z)$ which is analytic in some disc centered at 0 and which has the property that $\big(q(z)\big)^2 = z$ for all z in the disc?

14. What is the radius of convergence of the Taylor series centered at $z = i$ for the function $M(z) = \dfrac{\sin z}{z}$?