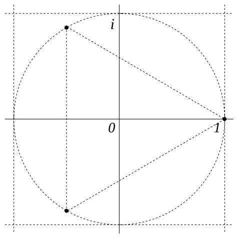
640:403:01 **Answers to the First Exam** **3/8/2001**

(8) 1. Find the principal argument $\operatorname{Arg} z$ and the modulus $|z|$ if $z = (2-2i)^7$.

Answer $2-2i$ is in the fourth quadrant, and one value of its argument is $-\frac{\pi}{4}$. One value of the argument of its seventh power is $-\frac{7\pi}{4}$.The "Arg" of that number must be in the interval between $-\pi$ and π (including π), so $\operatorname{Arg} z$ is $\frac{\pi}{4}$. Also $|2-2i| = 2\sqrt{2} = 2^{3/2}$, so $|z| = |2-2i|^7 = 2^{21/2}$.
Alternatively, $z = 2\sqrt{2}e^{-i\pi/4}$ so $z^7 = 2^{21/2}e^{-7i\pi/4}$, leading to the same answer.

(12) 2. Describe all solutions of $z^3 = 1$ algebraically in [either] rectangular [or polar] form. Sketch these solutions on the axes provided. *The brackets indicate text that should have been deleted!*



Answer If $z^3 = 1$ and $z = re^{i\theta}$, then $z^3 = r^3e^{3i\theta}$. If $r^3 = 1$, with r a non-negative real, then $r = 1$. 3θ must be 0 (mod 2π), which leads to the list $z = e^{2\pi ik/3} = \cos(2\pi k/3) + i\sin(2\pi k/3)$ for integers k in the set $\{0, 1, 2\}$. These values of sine and cosine are "well-known", so the roots are $\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\}$. The picture I want will indicate the vertices of a equilateral triangle inscribed in the unit circle with one vertex at 1. The central angle between the vertices is $2\pi/3$ or $120°$.

(14) 3. Suppose f is an analytic function with real part $u(x,y)$ and imaginary part $v(x,y)$. Explain why the function $H(x,y) = \big(u(x,y)\big)^2 - \big(v(x,y)\big)^2$ is harmonic, and find a harmonic conjugate of $H(x,y)$ in terms of $u(x,y)$ and $v(x,y)$.

Answer $Q(z) = z^2 = x^2 - y^2 + i(2xy)$ is analytic, and since composition of analytic functions is analytic, $Q(f(z))$ is analytic. The real part of $Q(f(z))$ is $H(x,y)$, and since the real part of an analytic function is harmonic, $H(x,y)$ is harmonic. The imaginary part of $Q(f(z))$ will be a harmonic conjugate of $H(x,y)$, and this is $2u(x,y)v(x,y)$.

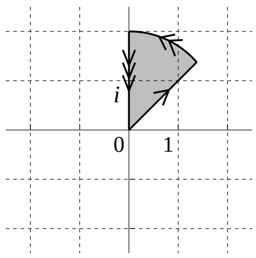
An answer can also be gotten through a direct computation of the Laplacian of H, use of the Cauchy-Riemann equations, etc., but the computation is elaborate, with many opportunities for error. A solution might go like this:

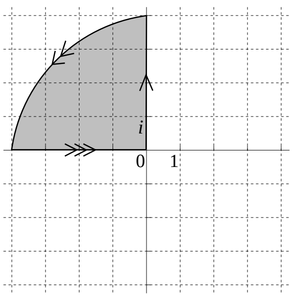
First derivatives
$$\begin{cases} H_x = 2uu_x - 2vv_x \\ H_y = 2uu_y - 2vv_y \end{cases}$$
Second derivatives
$$\begin{cases} H_{xx} = 2u_xu_x + 2uu_{xx} - 2v_xv_x - 2vv_{xx} \\ H_{yy} = 2u_yu_y + 2uu_{yy} - 2v_yv_y - 2vv_{yy} \end{cases}$$
Use of the Cauchy-Riemann equations
$$\begin{cases} H_{xx} = 2u_xu_x + 2uu_{xx} - 2u_yu_y - 2vv_{xx} \\ H_{yy} = 2u_yu_y + 2uu_{yy} - 2u_xu_x - 2vv_{yy} \end{cases}$$

Then add, and notice since u and v are both harmonic (so $u_{xx}+u_{yy} = 0$ and $v_{xx}+v_{yy} = 0$) that $H_{xx}+H_{yy} = 0$. Some further careful effort is needed to find a harmonic conjugate of H. If K is such a function, then $K_x = -H_y = -(2uu_y - 2vv_y)$ and $K_y = H_x = 2uu_x - 2vv_x$ so again using the Cauchy-Riemann equations, $K_x = -(-2uv_x - 2vu_x) = 2uv_x + 2vu_x$ and $K_y = 2uv_y - 2v(-u_y) = 2uv_y + 2vu_y$. An inspired guess (?) shows that $K = 2uv$ satisfies these equations.

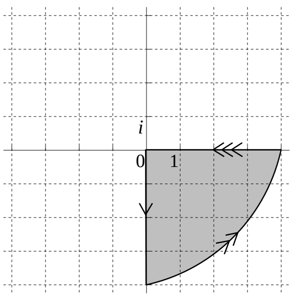
(12) 4. Sketch the region R of z's satisfying the inequalities: $0 < |z| < 2$ and $\frac{\pi}{4} < \operatorname{Arg} z < \frac{\pi}{2}$. What happens to R under the mapping $z \mapsto -z^2$? Sketch the image region carefully, and indicate what happens to the boundary of the region under the mapping. You may want to do this in two stages, first mapping R by $z \mapsto z^2 = w$ and then mapping $w \mapsto -w$.

Answer A sketch of R appears to the right with the edges labeled.





When $z \mapsto z^2 = w$, then $re^{i\theta} \mapsto r^2e^{2i\theta}$. Therefore central angles (relative to the origin) get doubled, and lengths to the origin get squared. The picture on the left is the result. The effect on the boundary has been indicated. When $w \mapsto -w$, the result is the picture on the right. Points get sent to antipodal points reflected through the origin. Again, the boundary effects have been specifically shown.



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(10) 5. Find the two points in the complex plane where the function $\left(\frac{1}{4}x^4 + y^2\right) + i(x^2y)$ is complex differentiable. Where is this function analytic?

Answer The function is complex differentiable where the Cauchy-Riemann equations are satisfied. Here $u(x, y) = \left(\frac{1}{4}x^4 + y^2\right)$ and $v(x, y) = x^2y$. $u_x = v_y$ becomes $x^3 = x^2$ and solutions of this are $x = 0$ and $x = 1$. $u_y = -v_x$ is $2y = -2xy$ here. When $x = 0$ we see that $y = 0$. When $x = 1$, $y = 0$. The two points are 0 and 1. The function is not complex differentiable in any open set so it is analytic nowhere.

(6) 6. Describe with some justification what happens to $\sin z$ as $z \to \infty$ along the positive imaginary axis.

Answer $\sin z = \sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$. When $x = 0$ the equation becomes $\sin(iy) = i\sinh(y)$. As $y \to +\infty$, $\sinh y$ increases steadily without any upper bound. So on the positive imaginary axis as $y \to +\infty$, $\sin z$ is imaginary, increasing, and unbounded.

(14) 7. Suppose $S(z)$ is the power series $\sum\limits_{n=1}^{\infty} \frac{z^n}{n(2+i)^n}$. a) What is the radius of convergence of $S(z)$?

Answer The Ratio Test may tell where the series converges absolutely (and where it diverges). So if the n^{th} term is $\frac{z^n}{n(2+i)^n}$, we must consider the ratio $\left|\frac{\frac{z^{n+1}}{(n+1)(2+i)^{n+1}}}{\frac{z^n}{n(2+i)^n}}\right| = \left|\frac{n}{n+1} \cdot \frac{z}{2+i}\right|$. As $n \to \infty$ the first factor $\to 1$ and the limit of the ratio is $\frac{|z|}{|2+i|}$. The Ratio Test can be usefully applied: the series converges (even absolutely) if $|z| < |2 + i|$ and diverges if $|z| > |2 + i|$. The radius of convergence is $|2 + i| = \sqrt{5}$.

b) Write a power series representing $S'(z)$ inside the radius of convergence of $S(z)$. Find the exact value of $S'(1)$. **Answer** Since $S(z) = \sum\limits_{n=1}^{\infty} \frac{z^n}{n(2+i)^n}$, $S'(z) = \sum\limits_{n=1}^{\infty} \frac{z^{n-1}}{(2+i)^n}$ inside the radius of convergence. Since $1 < \sqrt{5}$, 1 is inside the radius of convergence and $S'(1)$ must be $\sum\limits_{n=1}^{\infty} \frac{1}{(2+i)^n}$. This is a geometric series with first term $a = \frac{1}{2+i}$ and ratio $r = \frac{1}{2+i}$, so the sum is $\frac{a}{1-r} = \frac{\frac{1}{2+i}}{1-\frac{1}{2+i}} = \frac{1}{(2+i)-1} = \frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i$.

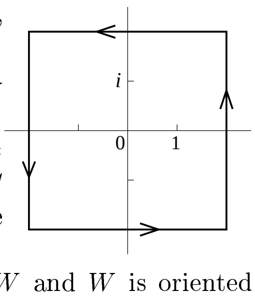
(16) 8. a) Compute $\displaystyle\int_C (x^2 + iy)\, dz$ where C is the straight line segment from 0 to $1 + 2i$.

Answer Parameterize by $z = x + iy = (1 + 2i)t$ with $0 \le t \le 1$. Then $dz = (1 + 2i)\, dt$ and $x = t$ and $y = 2t$, so that $x^2 + iy = t^2 + 2it$. The contour integral becomes $\displaystyle\int_0^1 (t^2 + 2it)(1 + 2i)\, dt = \left(\frac{1}{3} + i\right)(1 + 2i)$. This answer is fine. If you must "simplify", you should get $-\frac{5}{3} + \frac{5}{3}i$.

b) If C is the circle of radius 2 centered at 0, verify that $\left|\displaystyle\int_C \left(\frac{z^2 + 3}{5z^3 - 3}\right) dz\right| \le \frac{28\pi}{37}$. Show details of your estimates.

Answer We overestimate the modulus of the integral by ML where L, the length, is (2π)radius $= (2\pi)2 = 4\pi$, and M is an overestimate of the integrand on the circle. The integrand is a quotient. We estimate it by first overestimating the top: $|z^2 + 3| \overset{\triangle}{\le} |z|^2 + 3 = 2^2 + 3 = 7$ on the circle. We next underestimate the bottom: $|5z^3 - 3| = |5z^3 + (-3)| \underset{\nabla}{\ge} |5z^3| - |-3| = 5|z|^3 - 3 = 5 \cdot 2^3 - 3 = 37$ on the circle. Therefore the modulus of the integrand is estimated by $\frac{7}{37}$ and $ML = \frac{7}{37} \cdot 4\pi = \frac{28\pi}{37}$ as desired.

(10) 9. Compute $\int_W 5z^8 - \frac{2}{z}\, dz$ where W is the boundary of the square with vertices $-2-2i$, $2 - 2i$, $2 + 2i$, and $-2 + 2i$ as shown.



Answer Since integration is linear, we can compute $\int_W 5z^8\, dz$ and $\int_W -\frac{2}{z}\, dz$ separately and add the results.

If n is a positive integer, then (as shown in the text and discussed in class) $\int_C z^n\, dz = \frac{1}{n+1}(\text{Head of } C)^{n+1} - \frac{1}{n+1}(\text{Tail of } C)^{n+1}$ where C is any piecewise smooth curve. If C is a closed curve, then (since Head of C = Tail of C) the integral must be 0. Therefore $\int_W 5z^8\, dz = 5\int_W z^8\, dz = 0$.

We saw in class (and read in the textbook) that $\int_W \frac{1}{z}\, dz = 2\pi i$ since 0 is inside W and W is oriented positively (counterclockwise). Therefore $\int_W -\frac{2}{z}\, dz = -2\int_W \frac{1}{z}\, dz = -4\pi i$.

The final answer is $0 - 4\pi i = -4\pi i$.