February 27, 2002

Review Problems for the first exam in section 1 of Math 403

Most of these are from previous Math 403 exams, and some are from the text. Two previous "instantiations" of the course can be viewed on the web, linked from our course home page. They have review problems and exams, also. The exams generally should have answers displayed. The timing and topics of these courses was different from ours, especially since the text was different.

Elementary computations

1. Find the value(s) of the given expressions.

a) $\mathrm{Re}\left(\frac{1+i}{1-i}\right)$ b) $\mathrm{Im}\left(\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)^9\right)$ c) $\log(-1+i)$ d) $\mathrm{Log}(i)$ e) $\sin(i\,\mathrm{Log}(2))$

2. Is there any complex number z for which $\tan(z) = i$? If your answer is yes, find all such z. If your answer is no, justify your answer.

Basic series

3. Determine whether the given infinite series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2i}\right)^n$.

4. Describe the set of all points z in $\mathbb{C}$ for which the series

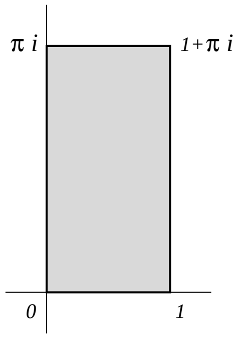
$$\sum_{n=2}^{\infty} \frac{z^n}{n^2-1}$$

converges. Justify your answer carefully.

Pictures

5. Sketch the region R of z's satisfying the inequalities: $\frac{1}{3} < |z| < 2$ and $\frac{\pi}{4} < \mathrm{Arg}\, z < \frac{\pi}{2}$. What happens to R under the mapping $z \mapsto \frac{1}{z}$? Sketch the image region carefully, and indicate what happens to the boundary of the region under the mapping.

6. Suppose S is the rectangular region whose boundary is the rectangle with corners 0, 1, πi, and $1+\pi i$ as shown. Sketch the image of S under the exponential mapping. Is the exponential mappying one-to-one on S?



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7. **More computations** Solve for all possible z. Give your answers in rectangular form:
a) $e^z = 1$ b) $\cos z = 0$ c) $\operatorname{Log} z = i$ d) $z^4 = 1 + i$ e) $\sin z = 2$

8. **Trigonometry** Verify the identity $\sin 2z = 2 \sin z \cos z$ for all complex numbers z starting from the exponential definitions of sine and cosine.

9. **Differentiability** a) Show that $g(z) = z\overline{z}$ is differentiable at 0 and is *not* differentiable at 1. In what set is g analytic?

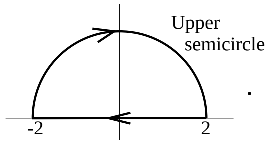
b) Suppose that u and v are the real and imaginary parts of an analytic function, h. Suppose that the range of h always lies on the hyperbola, $uv = 1$. Prove that h must be constant.

10. **Harmonicity** a) Select constants A and B and C so that $x^3 + Ax^2y + Bxy^2 + Cy^3$ is harmonic, and find all harmonic conjugates of this function.

b) Verify that $\arctan\left(\dfrac{2xy}{x^2 - y^2}\right)$ is harmonic*.

11. **Complex line integrals** Compute:

a) $\int_C (2x - iy)\, dz$ where C is the curve $y = x^2$ for $0 \le x \le 2$.

b) $\int_C \overline{z}\, dz$ where C is 

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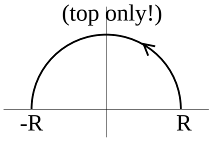
c) $\int_C z^{13} + \sin z\, dz$ where C is some curve from 1 to $\pi + i$.

12. **Integral estimation** a) Suppose C_R is the circle $|z| = R$ $(R > 1)$, described in the counterclockwise fashion. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| \le 2\pi \left(\frac{\pi + \ln R}{R} \right).$$

b) Show that $\lim\limits_{R \to \infty} \displaystyle\int_{C_R} \frac{z^2 - 1}{z^4 + 1}\, dz = 0$ where C_R is 

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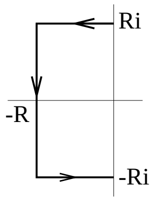
c) Show that $\lim\limits_{R \to \infty} \displaystyle\int_{C_R} \frac{e^z}{z^2 - 1}\, dz = 0$ where C_R is 

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* Please don't do a direct computation! Try to recognize where the pieces come from.