Answers to the Math 503 Midterm Exam

1. Parameterize the integral with $\theta \to e^{i\theta}$. Then $I(f,s) = is\int_0^{2\pi} f(se^{i\theta})e^{i\theta}\,d\theta$. Since we know that the limit of a product is equal to the product of the limits if the limit of each factor exists we only consider the non-trivial factor $F(s) = \int_0^{2\pi} f(se^{i\theta})e^{i\theta}\,d\theta$. Now $|F(s) - F(1)| \leq (2\pi)\sup\{|f(se^{i\theta}) - f(e^{i\theta})| : \theta \in [0, 2\pi]\}$ using the standard estimate for integrals and that $|e^{i\theta}| = 1$ when θ is real. But f is continuous on $\mathbb{C}$ and hence uniformly continuous on any compact set, e.g., $K = \overline{\Delta}(0,2)$. So given $\epsilon > 0$, there is $\delta > 0$ so that if z_1 and z_2 are in K with $|z_1 - z_2| < \delta$ then $|f(z_2) - f(z_2)| < \epsilon$. Therefore if $|s-1| < \min(1,\delta)$, $|f(se^{i\theta}) - f(e^{i\theta}| < \epsilon$ for any real θ. We see that $|F(s) - F(1)| < 2\pi\epsilon$, so that $\lim\limits_{s\to 1} F(s) = F(1)$ as asked.

2. a) $f'(z) = \frac{1}{2\pi i}\int_{\Delta(z,1)} \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta$ so $|f'(z)| \leq \frac{2\pi}{2\pi}\cdot\sup\{|f(\zeta)| : |\zeta - z| = 1\}$. When the given estimate works, $|f(\zeta)| \leq Ce^{(C|\zeta|^A)} \leq Ce^{(C(|z|+1)^A)} \leq Ce^{(C2^A|z|^A)}$. The last inequality is valid if $|z| \geq 1$. We have therefore shown an inequality of the desired type for f', with the same A as needed for f. The "new" C seems to be $\max\{C+1, 1, C2^A\}$. $C+1$ comes from the need to apply f's estimate in the Cauchy Integral Formula for f', 1 is needed as described above, and the last number is needed for the exponential bound.
b) If $f(x) = \sin(e^{(e^x)})$, then f satisfies one of the inequalities with $C = 1$ and $A = 0$ but $f'(x) = \cos(e^{(e^x)})\cdot e^{(e^x)}\cdot e^x$ does not since the cosine part is 1 infinitely often as $x \to +\infty$, and $e^{(e^x)} \leq Ce^{(Cx^A)}$ is false for *any* choices of C and A when x is sufficiently large.

Comment Functions satisfying such inequalities arise in constructing fundamental solutions of partial differential equations using Fourier transforms.

3. Let $A = \{z \in \mathbb{C} : r < |z| < R\}$. $\sup\{|p(z) - z^{-1}| : z \in A\} = \sup\{\frac{1}{|z|}|zp(z) - 1| : z \in A\} > \frac{1}{R}\sup\{|zp(z) - 1|; z \in A\}$. But $f(z) = zp(z) - 1$ is entire. The Maximum Modulus Theorem applies to f and $\overline{\Delta}(0,T)$. The sup of $|f(z)|$ on the boundary of that region must be at least $|f(0)| = 1$. The boundary of that disc is in A for any T between r and R. Thus we can take ϵ equal to $\frac{1}{R}$.

Alternatively Consider $\int_{\partial\Delta(0,T)} p(z) - z^{-1}\,dz$. Its value is $-2\pi i$ because $p(z)$ is entire. The standard estimate yields $2\pi \leq 2\pi T\sup\{|p(z) - z^{-1}| : |z| = T\}$. If $r < T < R$ then $T^{-1} \leq \sup\{|p(z) - z^{-1}| : z \in A\}$. The inf of the left-hand side is $\frac{1}{R}$.

More alternatively The value of $|z^{-1}|$ on $|z| = r$ is r^{-1} and on $|z| = R$ is R^{-1}. If there is *no* ϵ as indicated, there is a polynomial $p(z)$ with $\sup\{|p(z) - z^{-1}| : z \in A\} < \frac{1}{mR}$ (where $m > 1$ will be chosen later). By continuity, $\sup\{|p(z) - z^{-1}| : z \in \overline{A}\} \leq \frac{1}{mR}$. But then $\sup\{|p(z)| : z \in \partial\Delta(0,R)\} > \frac{1}{R} - \frac{1}{mR} = \frac{m-1}{mR}$ and $\sup\{|p(z)| : z \in \partial\Delta(0,r)\} < \frac{1}{r} + \frac{1}{mR}$. But if m is chosen so that $\frac{m-1}{mR} > \frac{1}{r} + \frac{1}{mR}$ (possible since $\frac{1}{R} > \frac{1}{r}$) we have contradicted the Maximum Modulus Theorem for $p(z)$. (Idea contributed by a student.)

Comment Compare this result with the Weierstrass Approximation Theorem in real variables. Polynomials in z *cannot* uniformly approximate z^{-1} on any set surrounding 0.

4. If $u = \operatorname{Re} f$ then u is continuous. u must be bounded, because the limit statement implies that $\exists M > 0$ so that $|u(z)| < 1$ for $|z| > M$ and u is certainly bounded on $\overline{\Delta}(0,M)$. Therefore the image of the entire function f is contained in a half-plane (where $\operatorname{Re} z \leq u$'s bound, for example). But a half-plane is conformally equivalent to the unit disc, so a composition of f with such a mapping will result in a bounded entire function, which is constant by Liouville's Theorem. Applying the inverse of that mapping shows that f itself must be constant. Since $f(0) = 0$ we know f is always 0. **Alternatively** Consider e^f so that $|e^f| = e^u$ is bounded.

5. A partial fraction decomposition is useful: $\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)$.
a) Use $\mathfrak{Cauchy}$ (= the Cauchy Integral Formula + Cauchy's Theorem) for $\partial\Delta(1,1) \cup \partial\Delta(-1,1)$ to get $J(a,b) = \left(\frac{2\pi i}{a-b}\right)\cdot(e^a - e^b)$. If h is $a - b$, $J(a,b)$ becomes $(2\pi i)\cdot\left(\frac{e^h - 1}{h}\right)\cdot e^b$. We then have the limit of a product (as a & $b \to 0$, equivalently, as h & $b \to 0$). Since the limits of the factors exist (one is the derivative of the exponential function at 0 and one is just the exponential function at 0), the desired limit must be $2\pi i$.
b) $\mathfrak{Cauchy}$ again applies directly, but the $\frac{1}{z-a}$ part of h is now holomorphic near $\overline{\Delta}(1,1) \cup \overline{\Delta}(-1,1)$ and yields 0 in the integral. $J(a,b)$ becomes $\left(\frac{2\pi i}{a-b}\right)\cdot(-e^b)$. This limit is ∞^* since we can make $|2\pi i e^b|$ close to 2π by choosing b small ($|b| < \rho$, say), and if we then take $a \in \Delta(0,\rho) \cap \overline{\Delta}(1,1)^c$, we see that $|J(a,b)| > \frac{\pi}{\rho}$.
c) Again use $\mathfrak{Cauchy}$ (or argue with winding number). The two parts of the integrand each contribute 0 to one of the two parts of the integral. The resulting value of $J(a,b)$ is $\left(\frac{2\pi i}{a-b}\right)\cdot(-e^a - e^b)$. The sign change results from the fact that $\partial\Delta(-1,1)$ is clockwise in Γ. Here also the limit is ∞, and the argument is close to that in b): the top approaches the constant $(2\pi i)\cdot(-2)$, and the bottom approaches 0.

*Most people would agree that this limit should be called "∞" in the "extended complex numbers" because it is $1/$(something whose limit is 0).