Some math stories with a complex moral

Power series

⟨*A standard topic in elementary real analysis*⟩

Here's a problem which could be analyzed in a beginning calculus course. Begin with a real power series: $\sum_{n=0}^{\infty} a_n x^n$, so all the a_n's are real. Suppose also that this series converges for $x = R$, which for simplicity we assume is positive. Since convergent series must have terms which approach 0, we know that $M = \sup\{|a_n|R^n, n \geq 0\}$ is finite. If we select x_0 closer to 0 than R (so $0 < x_0 < R$), we see that $|a_n x_0^n| = |a_n|R^n \left(\frac{x_0}{R}\right)^n \leq Mt^n$ where $t = \frac{x_0}{R}$ is less than 1. So $\sum_{n=0}^{\infty} a_n x_0^n$ converges absolutely and thus converges. If $x \in [-x_0, x_0]$, then $\sum_{n=0}^{\infty} a_n x^n$ converges also. In fact, it converges absolutely, because $\left|\sum_{n=0}^{\infty} a_n x^n\right| \leq \sum_{n=0}^{\infty} |a_n||x^n| \leq \sum_{n=0}^{\infty} |a_n||x_0^n|$ and we're done because of the comparison test. I'll write $f(x)$ for the sum of the series $\sum_{n=0}^{\infty} a_n x^n$ whenever it converges.

Now compare $f(x_1)$ and $f(x_2)$ when x_1 and x_2 are both in $[-x_0, x_0]$. Because $\sum_{n=0}^{\infty} |a_n|x_0^n$ converges, given $\epsilon > 0$, there is $N > 0$ so that $\sum_{n=N}^{\infty} |a_n|x_0^n < \epsilon$. Fix N and notice that $f(x) = \mathrm{Poly}(x) + \mathrm{Tail}(x)$ where $\mathrm{Poly}(x) = \sum_{n=0}^{N-1} a_n x^n$ and $\mathrm{Tail}(x) = \sum_{n=N}^{\infty} a_n x^n$. Comparison and the triangle inequality show that $|\mathrm{Tail}(x)| < \epsilon$ for any $x \in [-x_0, x_0]$. Since x_1 and x_2 are both in that interval, we see $|f(x_1) - f(x_2)| \leq |\mathrm{Poly}(x_1) - \mathrm{Poly}(x_2)| + |\mathrm{Tail}(x_1) - \mathrm{Tail}(x_2)| \leq |\mathrm{Poly}(x_1) - \mathrm{Poly}(x_2)| + |\mathrm{Tail}(x_1)| + |\mathrm{Tail}(x_2)|$. Since $\mathrm{Poly}(x)$ is a polynomial and polynomials are continuous and therefore uniformly continuous on closed, bounded intervals, there is $\delta > 0$ so that if $|x_1 - x_2| < \delta$ (with $x_1, x_2 \in [-x_0, x_0]$) then $|\mathrm{Poly}(x_1) - \mathrm{Poly}(x_2)| < \epsilon$.

We've shown that given $\epsilon > 0$, there is $\delta > 0$ so that if $|x_1 - x_2| < \delta$ and $x_1, x_2 \in [-x_0, x_0]$, then $|f(x_1) - f(x_2)| < 3\epsilon$. Inside its interval of convergence, the sum of a power series is a continuous function.

Further points to ponder

i) Give an example of a series of continuous functions which converge at every point of an interval to a function which is *not* continuous at every point of the interval.

ii) The polynomial partial sums converge *uniformly* to the sum of the infinite series in any closed and bounded interval inside the interval of convergence. Show by example that the partial sums need not converge uniformly in the whole interval of convergence.

Harmonic functions

⟨*Used in classical applied mathematics, differential equations, and probability*⟩

Here we need several variable calculus. Consider functions on $\mathbb{R}^2$ whose partial derivatives up to second order exist and are continuous: C^2 functions. We define the Laplacian, $\triangle$, a partial differential operator whose effect on C^2 functions is given by the formula $\triangle f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$. $\triangle$ is important in many models of physical phenomena. For example, solutions of $\triangle f = 0$ (which are called *harmonic functions*) occur as temperature distributions for two-dimensional steady state heat flow. They also appear in numerous other areas of mathematical physics, analysis, and probability.

Harmonic functions have an important *Mean Value Property*:

A harmonic function's average value over a circle equals its value at the circle's center.

Consider the case where the circle, C_r, has radius r and center $(0,0)$. The average value of f over C_r is $\displaystyle\int_{C_r} f(x,y)\, ds$ divided by $2\pi r$. Parameterize the circle: $\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \end{cases}$, with $0 \le \theta \le 2\pi$. Then $\begin{cases} dx = & -r\sin\theta\, d\theta \\ dy = & r\cos\theta\, d\theta \end{cases}$ and $ds = \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2}\, d\theta = r\, d\theta$.

Let's look at how the average value function $I_r = \dfrac{1}{2\pi r}\displaystyle\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\ r\ d\theta = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, d\theta$ varies with r. We know $I_0 = f(0,0)$ since when $r = 0$ we are integrating the constant $f(0,0)$. We compute $\dfrac{d}{dr} I_r$ to see how I_r changes. The several variable chain rule and Leibniz's Theorem about differentiation under the integral yield the following expression for this derivative:

$$\frac{1}{2\pi}\int_0^{2\pi} (f_x \cos\theta + f_y \sin\theta)\, d\theta = \frac{1}{2\pi r}\int_0^{2\pi} (f_x r\cos\theta + f_y r\sin\theta)\, d\theta = \frac{1}{2\pi r}\int_{C_r} f_x\, dy - f_y\, dx$$

The last equality follows from the "dictionary" of dx and dy we wrote above.

We can apply Green's Theorem to this line integral. Green's Theorem says that if Q is a nice curve bounding a region W in the plane, then

$$\int_Q M\, dx + N\, dy = \int\!\!\int_W N_x - M_y\, dA$$

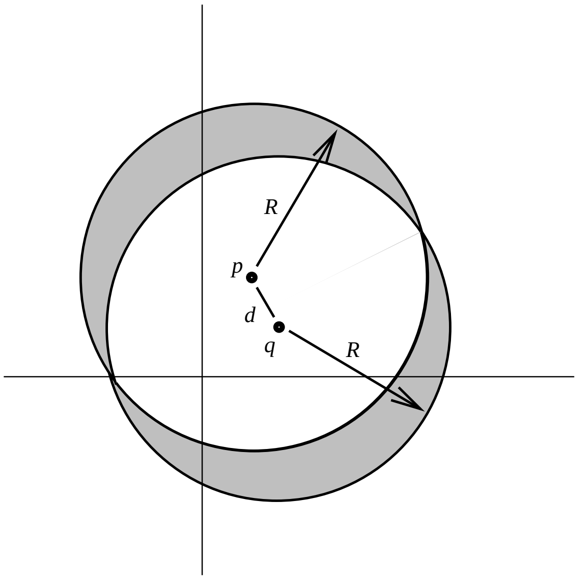
In our computation, Q is C_r (bounding a disc), M is $-f_y$, and N is f_x. Then $N_x - M_y = f_{xx} - (-f_{yy}) = \triangle f$ which is 0 when f is harmonic. I_r must be constant because it has derivative 0. Since I_0 is f's value at the center of the circle, I_r must always be $f(0,0)$. We proved the Mean Value Property.

Multiply the equation $f(0,0) = \dfrac{1}{2\pi r}\displaystyle\int_{C_r} f(x,y)\ ds$ by $2\pi r$ and integrate dr from 0 to R. We get $\displaystyle\int_0^R 2\pi r f(0,0)\ dr = \int_0^R \int_{C_r} f(x,y)\ ds\ dr$. This gives $\pi R^2 f(0,0) =$

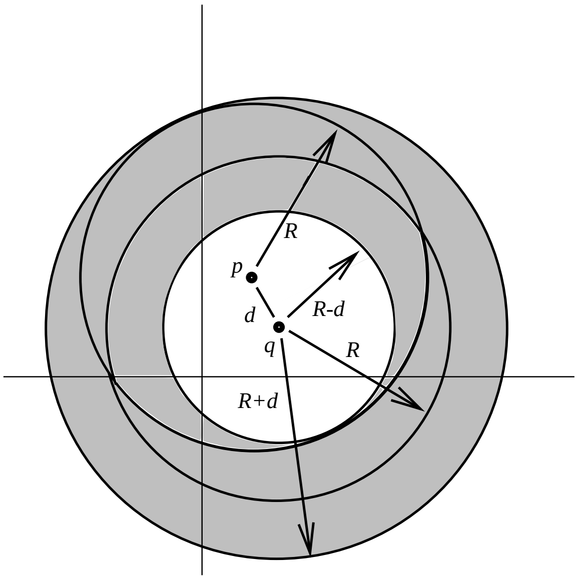
$\displaystyle\iint_{\overline{\Delta(0,R)}} f(x,y)\,dA$ where $\overline{\Delta(0,R)}$ is the closure of the open disc with center $(0,0)$ and radius R. This is the *Area Mean Value Property*:

A harmonic function's value at the center of a disc equals its area average over the disc.

Now suppose that f is harmonic on all of $\mathbb{R}^2$ and bounded: there is $B \geq 0$ with $|f(x,y)| \leq B$ for all $(x,y) \in \mathbb{R}^2$. We'll compare $f(p)$ and $f(q)$ cleverly for p and q in $\mathbb{R}^2$. Since we can replace $f(p)$ and $f(q)$ by their area mean values over discs of radius R centered at p and q respectively (this is in each case the double integral of f over the designated area divided by πR^2), the difference between them is overestimated by the integral of $2B$ over the two "lunes" which are the symmetric difference of the two discs, divided by πR^2.



The two lunes



The annular region

This integral can in turn be overestimated by $2B$ times the area of an annular region between two concentric circles: an inner circle with radius $R-d$ and an outer one with radius $R+d$ (here d is the distance between p and q). Therefore $|f(p)-f(q)| \leq \dfrac{1}{\pi R^2} 2B\left(\pi(R+d)^2 - \pi(R-d)^2\right) = \dfrac{8Bd}{R}$. We conclude that $|f(p)-f(q)|$ is 0 by letting $R \to \infty$. A bounded harmonic function defined on $\mathbb{R}^2$ must be constant.

Further points to ponder

i) Find a solution in $\mathbb{R}^2$ to the two-dimensional wave equation $\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial y^2} = 0$ which is bounded and non-constant. (Solutions of $\triangle f = 0$ and the wave equation are very different.)

ii) Find a harmonic function, f, which has $f_x = 3x - 5xy$. Verify that no harmonic function has $f_x = 3x^2 - 5y$.

iii) Show that a harmonic function defined in all of $\mathbb{R}^2$ can't be bigger (hotter!) at the origin than everywhere else: modeled steady state heat can't "pile up" or focus at a point.

The Fundamental Theorem of Algebra

$\langle$*Algebra intertwined with topology*$\rangle$

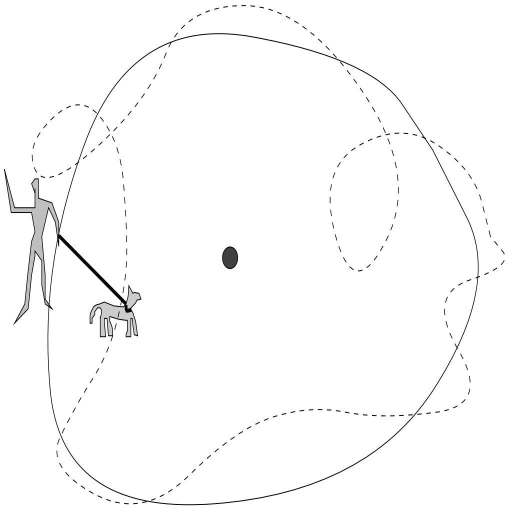
The complex numbers, $\mathbb{C}$, and the two-dimensional real vector space, $\mathbb{R}^2$, are the same. The complex number $z = x + yi$ corresponds to the vector (x, y). x is called the *real part* of z, Re z, and y is called the *imaginary part* of z, Im z. $\mathbb{C}$ is a field with addition and multiplication. If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ then $z_1 + z_2$ is, of course, the complex number corresponding to the vector sum (in $\mathbb{R}^2$) of two pairs of real numbers: it is $(x_1 + x_2) + (y_1 + y_2)i$. Since $i^2 = -1$ and the field axioms (including distributivity) should be true, the product of z_1 and z_2 must be $(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i$. Verification of the field axioms is direct and tedious. Understanding complex multiplication is easier if the corresponding points in $\mathbb{R}^2$ are described using polar coordinates. Assume we know θ_1 and θ_2 so that $\begin{cases} x_1 = & r_1 \cos\theta_1 \\ y_1 = & r_1 \cos\theta_2 \end{cases}$ and $\begin{cases} x_2 = & r_2 \cos\theta_2 \\ y_2 = & r_2 \cos\theta_2 \end{cases}$ for some nonnegative real numbers r_1 and r_2. Then $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.*. The product z_1z_2 can be written

$$r_1r_2\Big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\Big) = \\ r_1r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

The "r" part of the polar expression of a complex number, z, is called its *modulus*, denoted $|z|$, and the "θ" part is called its *argument*, denoted $\arg z$. The computation above shows that complex multiplication multiplies moduli and adds arguments. If N is a positive integer, the modulus of z^N is the N^{th} power of the modulus of z and the argument of z^N is N times the argument of z. Define $Q_N(z)$ to be z^N. As z travels once around a circle of radius R centered at 0 (starting and ending at $R + 0i$), $Q_N(z)$ will travel N full times around a circle of radius R^N centered at 0 (starting and ending at $R^N + 0i$).

Walking the dog

You're walking with your dog on a leash late at night on a city street. You want to keep within range of the street light, and you wind up walking around a lamppost. In order to avoid your dog and the leash getting tangled up with the lamppost, you always keep the leash shorter than your distance to the lamppost. If you walk completely around the lamppost N times, then your dog must also walk completely around the lamppost N times.



This statement can be proved by imagining a whole collection of "dog walks" parameterized by a real number s which is between 0 and 1. Each value of s creates a new path

* In some texts, $\cos\theta + i\sin\theta$ is abbreviated by cis θ.

for the dog to walk by multiplying the length of the dog's leash by a factor of s. When $s = 0$, you and the dog walk the identical path (maybe you are carrying the dog), and when $s = 1$ the dog walks its original path. s changes from 0 to 1 continuously. Since the dog never collides with the lamppost on any of these walks, and the integer number of times the dog walks around the lamppost is a continuous function of s, you and your dog indeed do walk around the lamppost the same number of times.

Let's apply the dog-walking logic to a monic polynomial of P of degree $N > 0$ acting on C_R, which again is the circle of radius R centered at the origin. R will be very large.

$$P(z) = z^N + \sum_{j=0}^{N-1} a_j z^j = Q_N(z) + L(z)$$

Here L stands for "lower order terms" or for "leash". Let's overestimate $|L(z)|$ using the triangle inequality ($|A + B| \leq |A| + |B|$) and the multiplicative behavior of moduli ($|AB| = |A|\,|B|$):*

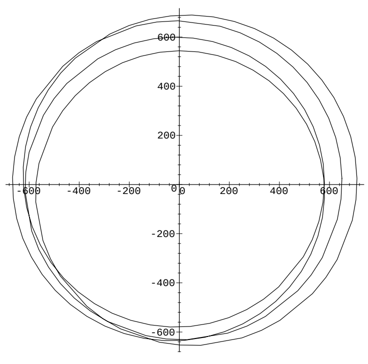
$$|L(z)| \leq \sum_{j=0}^{N-1} |a_j z^j| = \sum_{j=0}^{N-1} |a_j||z^j| \leq NT|z|^{N-1}$$

Here $T = \max\{|a_j| : 0 \leq j \leq N-1\}$. If now $R > NT$ and $z \in C_R$, we know

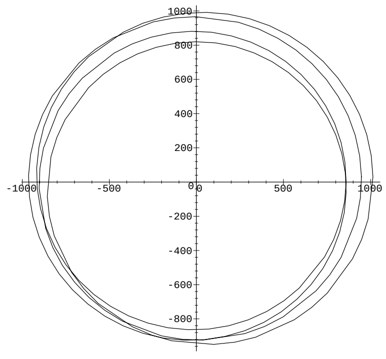
$$|Q_N(x)| = |z^N| = R|z|^{N-1} > NT|z|^{N-1} \geq |L(z)|$$

Therefore when R is large, the dog ($P(z)$) and you ($Q_N(z)$) walk around 0 the same number of times: N.

Note that as we change R, the image of C_R under the mapping $P(z)$ changes continuously. Here are some pictures of the image of C_R drawn by `Maple` when $P(z) = z^4 + (2+i)z^2 - 6z + 5$ for R large enough so that the lower order terms don't matter too much:



$R = 5$



$R = 5.5$

Examine the pictures carefully. Each picture is 4 (because of z^4) approximate circles around 0. Look at the scaling. The radius in the left-hand picture is about $5^4 = 625$ and in the right-hand picture is close to $(5.5)^4 \approx 915$. The pictures are almost identical except for the different scaling.

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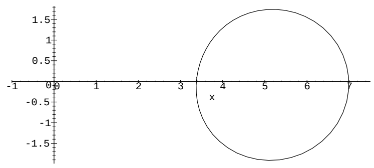
* We also need to make $|z| \geq 1$ so the moduli of higher powers of z don't decrease.

If $P(z)$ is *never* 0, then the image of C_R never goes *through* 0 and so it will always go around the origin the same integer number of times. We've seen already that if R is large, this number is N. But if R is very small, then $P(R\cos\theta + iR\sin\theta)$ is going to be very close to $P(0)$ for all θ. This is because polynomials are continuous and both sine and cosine are bounded by 1. But $P(0) \neq 0$, and for R really small, P's values on C_R are all at a distance less than $\frac{1}{2}|P(0)|$ from $P(0)$. The image of such C_R's under the mapping P *can't* go around 0 at all. So N (an integer not depending on R if P has no roots) must be 0! This is a contradiction, and therefore $P(z)$ must have a root.

We proved the *Fundamental Theorem of Algebra*. In fact, we've found a circle enclosing all the roots of a given polynomial.

Here's a picture if R is small of C_R's image under the same polynomial mapping, $P(z)=z^4+(2+i)z^2-6z+5$.



$R = .3$

Higher powers of z are negligible for small z. What's shown is almost the image of the circle of radius .3 under the mapping $z \to -6z+5$, which is a circle of radius 1.8 centered at 5.

Further points to ponder

i) Describe what the mapping $z = x + yi \to \overline{z} = x - yi$ (complex conjugation) does to a circle centered at the origin. Then conclude (paralleling the reasoning above) that any non-constant polynomial in $\overline{z}$ alone must have a root. Of course, you could also see this by conjugating the polynomial and getting one in z!

ii) Find a non-constant polynomial in both z and $\overline{z}$ which has *no* roots.

iii) Find a class of polynomials in z and $\overline{z}$ to which you can apply the reasoning presented here. Conclude that each such polynomial must have a root.

Preserving angles

⟨*Two dimensional geometry: orientation, area, length, angle preservation*⟩

Geometry was classically described by Felix Klein as the study of classes of transformations or mappings which preserve certain geometric properties or quantities. The transformations considered here act on open subsets of $\mathbb{R}^2$. Topological transformations (homeomorphisms) can distort or warp such a region arbitrarily. The special transformations studied in two-dimensional geometry could preserve quantities such as orientation, area, length, and angle. An *isometry* is a transformation preserving distance between all pairs of points. The identity map is an orientation-preserving isometry, as is any rotation about the origin. Complex conjugation, which sends $z = x + yi$ to $\overline{z} = x - yi$, is an orientation-reversing isometry. Geometrically it flips the point corresponding to a complex number over the x-axis, negating its argument and leaving its modulus unchanged. There are orientation-preserving maps leaving area unchanged which are not isometries, such as the linear mapping S taking (x, y) to $(2x, \frac{1}{2}y)$. Mappings preserving angles are called *conformal*. The identity and all rotations about the origin are conformal linear maps which preserve orientation (sometimes called *directly conformal*). Complex conjugation is an orientation-reversing conformal map. The conformal maps considered here will almost always preserve orientation. The linear transformation S described above is not conformal.

It's not difficult to decide when a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ preserves orientation or angles or areas or distances. Non-linear mappings are more complicated. The non-linear mappings which we can analyze most directly are the differentiable ones.

Suppose U and V are open subsets of $\mathbb{R}^2$ and $F\colon U \to V$ is a differentiable function. Then $F(x,y) = (u(x,y), v(x,y))$ where u and v are differentiable functions of two variables. Many geometric questions about F can be settled by examining F's "infinitesimal" behavior – that is, consider the derivative of F at $p \in U$, $F'(p)$, which is a linear mapping from $\mathbb{R}^2$ to itself. F will "preserve orientation" if it preserves orientation infinitesimally. That's guaranteed if $F'(p)$ (the Jacobian of F at p, represented by a 2×2 matrix) always has positive determinant. And F will be conformal if $F'(p)$ is always a conformal linear mapping. A differentiable transformation F is conformal when the angle between two smooth intersecting curves in its domain is the same as the angle between the images under F of the two curves.

Consider two open sets in $\mathbb{R}^2$: the upper half-plane, $H = \{z = x + yi : \text{Im } z = y > 0\}$, and the unit disc $\Delta(0,1) = \{z = x + yi : |z| = \sqrt{x^2+y^2} < 1\}$. Certainly there is no transformation $F\colon H \to \Delta(0,1)$ which is isometric or preserves area using the standard Euclidean notions of area and length because H has infinite area and $\Delta(0,1)$ does not. It is surprising that there is a conformal equivalence between H and $\Delta(0,1)$. Here is one such mapping, described rather explicitly:

$$C(x,y) = \big(u(x,y), v(x,y)\big) \text{ where } \begin{cases} u(x,y) = \dfrac{x^2+y^2-1}{x^2+y^2+2y+1} \\ v(x,y) = \dfrac{-2x}{x^2+y^2+2y+1} \end{cases}$$

We must check that C maps H to $\Delta(0,1)$ and that every $C'(p)$ is conformal. This can be done using the methods of multivariable calculus, but the verification is easier with complex arithmetic, and may also provide some insight. The $u(x,y)$ and $v(x,y)$ defined above are actually the real and imaginary parts of a function from complex analysis called the *Cayley Transform*, defined by $C(z) = \dfrac{z-i}{z+i}$.

When is $|C(z)| < 1$? Since modulus works well with multiplication and division, the inequality $|C(z)| < 1$ is equivalent to $|z-i| < |z+i|$ (note that $z = -i$ isn't valid in either expression!). This in turn becomes $|z-i|^2 < |z+i|^2$, and by substituting $z = x + yi$ we get a statement written with only real variables:

$$x^2 + (y-1)^2 < x^2 + (y+1)^2$$

Further simplification yields $-2y < 2y$ or just $y > 0$. So $C(z) \in \Delta(0,1)$ is equivalent to $z \in H$.

Now suppose that $t \to x(t) + y(t)i$ is a smooth curve. Its tangent vector at $t = 0$ is $x'(0) + y'(0)i$. What is the tangent vector to the curve $t \to C\big(x(t) + y(t)i\big)$? We can find out by differentiating $C\big(x(t)+y(t)i\big) = \dfrac{x(t) + y(t)i - i}{x(t) + y(t)i + i}$. This is easy because the algebraic rules of differentiation are valid for complex-valued functions of one real variable.

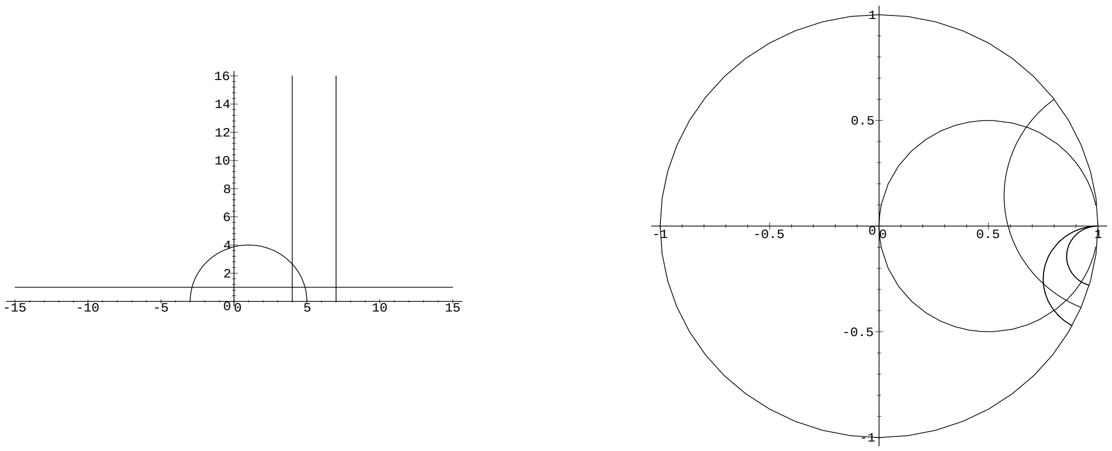
$$\frac{d}{dt}\left(C\big(x(t)+y(t)i\big)\right)=\frac{\big(x'(t)+y'(t)i\big)\big(x(t)+y(t)i-i\big)-\big(x'(t)+y'(t)i\big)\big(x(t)+y(t)i+i\big)}{\big(x(t)+y(t)i+i\big)^2}$$

Set $t=0$. The image of the tangent vector $x(0)+iy(0)$ is:

$$\frac{\big(x'(0)+y'(0)i\big)\big(x(0)+y(0)i-i\big)-\big(x'(0)+y'(0)i\big)\big(x(0)+y(0)i+i\big)}{\big(x(0)+y(0)i+i\big)^2}$$

The image is therefore obtained by multiplying $x'(0)+y'(0)i$ by the complex number $\dfrac{-2i}{\big(x(0)+y(0)i+i\big)^2}$. Of course if $x(0)+y(0)i\in H$ this number is defined and non-zero. Multiplication by a non-zero complex number rotates the corresponding vector by an angle equal to the argument of the complex number, as we saw before. Thus the derivative of C at every point is a conformal linear mapping. Every two smooth curves which intersect in H are mapped to two smooth curves which intersect in $\Delta(0,1)$, and tangent vectors to the curves at the intersection point both get rotated by the same amount. We have shown that H and $\Delta(0,1)$ are conformally diffeomorphic, using C and its inverse.

Below are `Maple` pictures of some curves in H and their images under C in $\Delta(0,1)$. Mappings like C provide such interesting geometric insight that new notions of area and length were invented. Using these definitions, C and other conformal mappings become isometries. H and $\Delta(0,1)$ are models of hyperbolic geometry, a classical non-Euclidean geometry with infinitely many "lines" parallel to a given "line" through an external point. This is the geometry of a complete surface with constant negative Gaussian curvature.



The curves in H are an upper semicircle of radius 4 and center (1,0), the vertical lines (really rays or half-lines) x=4 and x=7, and the horizontal line y=1. Angles are preserved as predicted. C changes elements of {lines}∪{circles} to other elements of {lines}∪{circles}. This is not obvious! The vertical lines and the half-circle are *geodesics* (curves minimizing length) in hyperbolic geometry. The pictures are easier to understand if you agree that $C(\infty)$ should be 1. Then the vertical and horizontal lines in H meet at ∞ so their images in Δ(0,1) will meet at 1.

Further points to ponder

Suppose $F(x,y) = (u(x,y), v(x,y))$ is a differentiable function from $\mathbb{R}^2$ to itself. Its derivative at a point is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, which can be represented by a 2×2 matrix: $\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$. Algebraic conditions on this matrix insure that the linear transformation has certain desirable geometric properties.

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation whose matrix with respect to the standard basis $\{(1,0),(0,1)\}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then:

i) T preserves orientation $\iff$ $\det T = ad - bc > 0$.
ii) T preserves area $\iff$ $|\det T| = 1$.
iii) T preserves length $\iff$ $a = \pm d$ and $b = \mp c$ and $|\det T| = 1$.
iv) T preserves angles between lines through the origin $\iff$ $T \neq 0$ and $a = \pm d$ and $b = \mp c$.

Thus, by i) and iv), a function F given as above will be directly conformal if and only if its derivative is never 0 and the *Cauchy-Riemann equations* are always true: $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$.

What's going on?

$\langle$*All part of something beautiful . . .* $\rangle$

We looked at:

- Local uniform convergence of series of functions.
- Line integrals and their estimation; Green's Theorem.
- Deformation of curves (homotopy); winding numbers.
- Angle-preservation under non-linear mappings.

All these ideas are important in complex analysis. They are even more powerful when considered together in the context of this beautiful subject. We will study the ideas presented here systematically and link them and others. We'll learn why one theme in mathematics is to "complexify" almost any function or any problem and then appreciate the resulting insight.

I showed nothing directly related to number theory since it's somewhat difficult to give a proof of the Prime Number Theorem in a few pages! Complex analysis is very important in number theory. Also, nothing was presented here directly relevant to "modern" applied mathematics: theoretical computer science and discrete mathematics. However, we can read in the introduction to *Mathematics for the Analysis of Algorithms* by Daniel Greene and Donald Knuth that ". . . it is assumed . . . the reader is familiar with the fundamentals of complex variable theory and combinatorial analysis." These authors believe complex analysis is vital for their subject.

We'd better get to work now.

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