A jump into the twentieth century

Preface

Almost everything in the course we're concluding could have been taught a century ago. Some points of view might seem strange to century-old sensibilities, but except for parts of the homology/homotopy versions of Cauchy's Theorem, the material likely would not have surprised Hadamard or Weierstrass or Riemann. They might even consider a few things we've done to be annoyingly and inessentially precise. But complex analysis has continued to grow and change in more fundamental ways.

Study of complex variables itself has become very technical. It wouldn't be easy to cite results in an understandable (or interesting!) way. To me the most excitement has occurred as geometry and partial differential equations have become involved. Really different approaches to one complex variable can be found in [K], using ideas from differential geometry, and in [H], using ideas from partial differential equations, or see [N], which has ideas from both. I will show you one significant result of twentieth century mathematics in this course. This result from partial differential equations uses complex analysis directly and essentially.

If $f = u + iv$, then f is holomorphic if $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$. There's another way of writing these equations. Since $\dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}$, the Cauchy-Riemann equations are equivalent to the real and imaginary parts of the following equation:

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

One-half of the left-hand side of this is called $\dfrac{\partial}{\partial \overline{z}}$ by complex analysts, since you get 1 if f is $\overline{z}$ and 0 if you feed in a "function of z alone and not of $\overline{z}$".

Solving differential equations

Differential equations should have solutions. This was impressed on me during a discussion with some physicists. Their logic was: differential equations describe physical situations and the physical situations exist, so the equations must have solutions.

• ODE's

Ordinary differential equations may describe quantities evolving through time: what happens to a rock thrown with initial vertical velocity v_0 from a building of height s_0 at time 0 on a planet whose gravitational attraction is a? One model asks us to solve an initial value problem:

$$\frac{d^2 s}{dt^2} = -a \text{ with } s(0) = s_0 \text{ and } v(0) = v_0$$

This differential equation has lots of solutions: $-\frac{1}{2}at^2 + (\text{Constant}_1)t + (\text{Constant}_2)$. The initial conditions allow us to specify a unique solution.

Now we can systematically study ODE's, guided by this example and others. But even with ODE's some care is needed. If the ODE is in standard form, such as $\frac{dy}{dx} = f(x,y)$ where f is a continuous (possibly complex-valued) function of two variables, then solutions must exist, but:

$\star$ *Solutions may not be unique.*

If $\frac{dy}{dx} = \sqrt{|y|}$, with $y(0) = 0$, then is one solution is $y(x) = 0$ for all x.

If $A > 0$, then $y(x) = \begin{cases} 0 & \text{if } x \le 2A \\ \left(\frac{1}{2}x - A\right)^2 & \text{if } x > 2A \end{cases}$ is also a solution.

$\star$ *Expect only local solutions.*

The equation $\frac{dy}{dx} = y^2$ with $y(0) = B > 0$ has solution $y = -\left(x - \frac{1}{B}\right)^{-1}$ with domain $\left(-\infty, \frac{1}{B}\right)$. Solutions may "blow up" shortly after 0.

Uniqueness is guaranteed if $f(x,y)$ satisfies a local Lipschitz condition in y which is true when f is C^1. Then theorems (see [CL]) verify that ODE's with reasonable f's do have local solutions uniquely specified by initial conditions.

• PDE's

Thrown rocks seem less intricate than sound waves and heat conduction, which partial differential equations are supposed to model. The first PDE which doesn't echo simple ODE behavior is probably the Laplacian, $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Since $\triangle(x^n y^m) = \left(n(n-1)y^2 + m(m-1)x^2\right)\left(x^{n-2}y^{m-2}\right)$, you can hope there is a process to stuff power series through $\triangle$ and get desired power series. There is such: the Cauchy-Kowalewski Theorem guarantees that PDE's with real analytic coefficients and real analytic right-hand sides ($Pf = g$, P and g all described real analytically) have local real analytic solutions. Other PDE's which come from physical models were studied extensively in the nineteenth century, and some generalizations were solved in the twentieth: $\triangle$ becomes what are called elliptic equations, and the wave and heat equations generalize to what are called hyperbolic and parabolic equations, respectively.

The initial conditions of ODE's are replaced by more complicated boundary value problems. For example, the maximum principle implies that specifying a harmonic function on the whole boundary of a bounded region guarantees uniqueness: this leads to the Dirichlet problem. Solutions for specific situations can then be written using complicated integral operators.

By the middle of this century, great efforts were made to obtain local existence results for general PDE's. Probably the most significant single result of the 1950's was showing that PDE's with constant coefficients can always be solved locally. Fourier transform methods with clever complex integration contours on the transform side were very important. The big remaining question was: can every PDE be solved?

The Lewy Equation

This linear partial differential equation was first studied by Hans Lewy in the late 1950's ([L1], [L2]):

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} + i2(x+iy)\frac{\partial f}{\partial s} = ig(s)$$

The function $f(x,y,s)$ is a complex-valued differentiable function in some neighborhood of the origin in $\mathbb{R}^3$, and $g(s)$ is a real C^∞ function of the one variable s in the same neighborhood. We will see that there are choices of g for which there is no solution f in *any* neighborhood of $(0,0,0)$.

The trick is to "project" f into a lower-variable situation and recognize the Cauchy-Riemann equation, and then take advantage of what is known about solutions of that equation (holomorphic functions). Here is the key ingenious definition:

$$F(r,s) = \sqrt{r}\int_0^{2\pi} e^{i\theta} f(\sqrt{r}\cos\theta, \sqrt{r}\sin\theta, s)\, d\theta$$

What differential equation will F satisfy? Note the change of variables from rectangular to "perturbed" polar: $\begin{cases} x = \sqrt{r}\cos\theta \\ y = \sqrt{r}\sin\theta \end{cases}$, so $x^2+y^2=r$. The function F is defined for r and s each close to 0 if also r is positive. We compute $\dfrac{\partial F}{\partial r}$ for $r>0$:

$$\begin{aligned}\frac{\partial F}{\partial r} = \frac{1}{2\sqrt{r}}\int_0^{2\pi} & e^{i\theta} f(\sqrt{r}\cos\theta, \sqrt{r}\sin\theta, s)\, d\theta \; + \\ & \sqrt{r}\int_0^{2\pi} e^{i\theta}\left(\frac{\partial f}{\partial x}\left(\frac{1}{2\sqrt{r}}\right)\cos\theta + \frac{\partial f}{\partial y}\left(\frac{1}{2\sqrt{r}}\right)\sin\theta\right) d\theta\end{aligned}$$

Here we used the product rule and then moved the differentiation inside the integral.

Integrate the first term on the right-hand side by parts:

$$\begin{array}{ccccc}\displaystyle\int_0^{2\pi} e^{i\theta} f\, d\theta & = & (f)(-ie^{i\theta})\Big]_0^{2\pi} & + & i\displaystyle\int_0^{2\pi} e^{i\theta}\left(\frac{\partial f}{\partial x}\sqrt{r}(-\sin\theta) + \frac{\partial f}{\partial y}\sqrt{r}(\cos\theta)\right) d\theta \\ \displaystyle\int u\, dv & = & uv & - & \displaystyle\int v\, du\end{array}$$

where the parts are $\left.\begin{array}{c} u = f \\ dv = e^{i\theta}\, d\theta \end{array}\right\}\left\{\begin{array}{l} du = \dfrac{\partial f}{\partial \theta} d\theta = \left(\dfrac{\partial f}{\partial x}\sqrt{r}(-\sin\theta) + \dfrac{\partial f}{\partial y}\sqrt{r}(\cos\theta)\right) d\theta \\ v = \dfrac{1}{i}e^{i\theta} = -ie^{i\theta}\end{array}\right.$

The boundary term $(f)(-ie^{i\theta})\Big]_0^{2\pi}$ is 0 because both $e^{i\theta}$ and $f(\sqrt{r}\cos\theta, \sqrt{r}\sin\theta, s)$ are periodic with period 2π.

Now reassemble $\dfrac{\partial F}{\partial r}$. All of the $\sqrt{r}$'s cancel.

$$\frac{\partial F}{\partial r} = \frac{i}{2}\int_0^{2\pi} e^{i\theta}\left(\frac{\partial f}{\partial x}(-\sin\theta) + \frac{\partial f}{\partial y}(\cos\theta)\right) d\theta + \frac{1}{2}\int_0^{2\pi} e^{i\theta}\left(\frac{\partial f}{\partial x}(\cos\theta) + \frac{\partial f}{\partial y}(\sin\theta)\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} e^{i\theta}\left(\frac{\partial f}{\partial x}(\overbrace{\cos\theta - i\sin\theta}^{e^{-i\theta}}) + \frac{\partial f}{\partial y}(\overbrace{\sin\theta + i\cos\theta}^{ie^{-i\theta}})\right) d\theta = \frac{1}{2}\int_0^{2\pi}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\theta$$

Of course $\dfrac{\partial F}{\partial s} = \sqrt{r}\displaystyle\int_0^{2\pi} e^{i\theta}\frac{\partial f}{\partial s}\, d\theta$. This is $\displaystyle\int_0^{2\pi}(x+iy)\frac{\partial f}{\partial s}\, d\theta$ because of our choice of perturbed polar coordinates. So $\dfrac{\partial F}{\partial r} + i\dfrac{\partial F}{\partial s} = \displaystyle\int_0^{2\pi}\left(\frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} + i(x+iy)\frac{\partial f}{\partial s}\right) d\theta = \int_0^{2\pi}\frac{1}{2}ig(s)\, d\theta = \pi i g(s)$.

So far F is defined on $(0,\delta)\times(-\delta,\delta)$ for $(r,s)\in\mathbb{R}^2$. We can shrink δ to insure that f is bounded since any continuous function is bounded in some neighborhood of every point of its domain. Because the definition of F has $\sqrt{r}$ outside the integral, we know that $\lim_{r\to 0+} F(r,s) = 0$. Finally, suppose that $\pi g(s) = h'(s)$ (that is, h is *some* antiderivative of πg) and put $G(r,s) = F(r,s) - h(s)$. Then

$$\frac{\partial G}{\partial r} + i\frac{\partial G}{\partial s} = \left(\frac{\partial F}{\partial r} + i\frac{\partial F}{\partial s}\right) - i(\pi g(s)) = \pi i g(s) - i(\pi g(s)) = 0$$

We have created a holomorphic function, G, whose domain is $(0,\delta)\times(-\delta,\delta)$. As $r\to 0^+$, $G(r,s)\to h(s)\in\mathbb{R}$: G has continuous *real* boundary values on a segment of the $r-$axis. The Schwarz reflection principle implies that G can be extended to a holomorphic function in $(-\delta,\delta)\times(-\delta,\delta)$. Here $G(r,s)$ would be defined by $\overline{G(-r,s)}$ for $r<0$ and $G(0,s)$ would just be $h(s)$.

But then h must be real analytic since it is the restriction of a holomorphic function to an open subset of one of the coordinate axes. Its derivative must also be real analytic. Thus, if the Lewy equation has a solution in some neighborhood of the origin in $\mathbb{R}^3$, the function g must be real analytic in some neighborhood of 0 in $\mathbb{R}$. This need not be true*. We have proved our assertion: the Lewy equation may have *no* solution in *any* neighborhood of the origin for certain C^∞ g's. Lewy's equation still has no solution even if the idea of solution is extended in various complicated ways (to distributions or hyperfunctions or ...). [A] shows that there are no distribution solutions in a manner similar to what's here, but both expositions are close to the original discussion in [L2].

Lewy's example was not anticipated! See the appendix, please.

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* For example, consider $g(s) = \begin{cases} e^{-1/s^2} & \text{if } s\neq 0 \\ 0 & \text{if } s=0 \end{cases}$.

Bibliography

[A] S. Agmon, *Lectures on Elliptic Boundary Problems*, Van Nostrand, Princeton, 1965. There are many good references on PDE's, for example, texts by Hörmander and by Treves. Go to the library.

[CL] E. Coddington and N. Levinson, *Theory of Ordinary Differential equations*, McGraw-Hill, New York, 1955. There are many good references on ODE's. Go to the library.

[L1] H. Lewy, On the local character of the solutions of an atypical linear partial differential equation in three variables and a related theorem for regular functions of two complex variables, *Ann. Math.* 64(1956), 514-522. Available on the web through the Rutgers library system.

[L2] H. Lewy, An example of a smooth linear partial differential equation without solution, *Ann. Math.* 66(1957), 155-158. Available on the web through the Rutgers library system.

[H] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Elsevier/ North-Holland, New York, 1990.

[K] S. Krantz, *Complex Analysis: The Geometric Viewpoint*, MAA, Washington, 1990.

[N] R. Narasimhan, *Complex Analysis in One Variable*, Birkhäuser, Boston, 1985.

Appendix

Here's the entire introduction to [L2], a paper with *no* bibliography – which is rather startling in an academic journal. Also realize that [L1]'s title contains a historic joke: the "atypical" partial differential equations without solutions turn out to be the collection of almost all equations!

> *In dealing with the existence of solutions of partial differential equations it was customary during the nineteenth century and it still is today in many applications, to appeal to the theorem of Cauchy-Kowalewski which guarantees the existence of analytic solutions for analytic partial differential equations. On the other hand a deeper understanding of the nature of solutions requires the admission of non-analytic functions in equations and solutions. For large classes of equations this extension of the range of equation and solution has been carried out since the beginning of this century. In particular much attention has been given to linear partial differential equation and systems of such. Uniformly the experience of the investigated types has shown that – speaking of existence in the local sense – there always were solutions, indeed, smooth solutions, provided that the equations were smooth enough. It was therefore a matter of considerable surprise to this author to discover that this inference is in general erroneous. More precisely, there exist linear partial differential equations with coefficients in C^∞ which possess not a single smooth solution in any neighborhood. The example to be presented in this paper is an equation of first order in three independent variables with complex-valued coefficients and unknown function, or, what amounts to the same, a system of two equations of first order for two functions of three variables, all real.*

Some further comments

These are in response to questions asked by several students to whom this material has been presented. Full answers to these questions would take months to describe.

How was the equation found? How was the analysis of the equation invented?

Lewy's example was discovered as part of a sequence of investigations which began towards the end of the nineteenth century. People wanted to learn how complex variables looked in more than one dimension. The traditional expression in English for this field of mathematics is *several complex variables.* Elementary results remain the same, but there is a distinctly different flavor even in some beginning examples, and there is behavior which has no analog at all in one complex variable (e.g., there are no isolated singularities in more than one dimension!).

Lewy wanted to investigate what happens to holomorphic functions in two variables restricted to the hypersurface Re $z = |w|^2$ for $(z, w) \in \mathbb{C}^2$. This hypersurface is one of the first non-planar examples one would look at, and therefore can serve as a simple model for seeing what bending a surface does. Phenomena from differential geometry and partial differential equations occur almost immediately. These make consideration of equations like Lewy's important, and they also supply suggestions for analyzing the equation. Thus the transition from f to F used in the proof is not random, but is intimately associated with the geometry of the hypersurface mentioned – a sort of quadratically distorted mean-value integral.

Is it just pathological behavior? Does it mean anything?

Almost all linear partial differential equations have this sort of "strange" behavior. The many equations solved in the nineteenth century and early twentieth century have much symmetry or many special properties. So, as mentioned before, far from being pathological, such equations as Lewy's are average!

Vector fields derived from geometry similar to the way Lewy got his equation have interesting algebraic properties. Study Lie algebras to learn more about this.

Topics related to the Lewy example turn out to be quite useful in fields outside of PDE, some of which purport to describe the "real world":

- Systems of partial differential equations derived from submanifolds of complex manifolds as the Lewy equation was have been studied by mathematical physicists to model particle interactions. Some knowledge of differential geometry is needed to understand their work.
- The real and imaginary parts of the Lewy operator provide two real vector fields in $\mathbb{R}^3$: $\dfrac{\partial}{\partial x} - 2y\dfrac{\partial}{\partial s}$ and $\dfrac{\partial}{\partial y} + 2x\dfrac{\partial}{\partial s}$. This is a smooth selection of two directions at every point. This pair of vector fields does not commute. The trajectories of the vector fields have interesting qualitative properties. Getting systems of vector fields this way is a source of useful examples for such areas of study as control theory.

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