October 20, 1997

Preparation for the midterm exam

The purpose of the exam is to assess your knowledge of complex variables and to prepare you for our written qualifying exam. The midterm exam will be given for two consecutive hours on Friday, October 24 with time and place to be announced in class. If you can't be there, please arrange an alternate time and place with me as soon as possible. The exam will consist of at most 5 questions. At least 2 of the 5 questions will be taken from the questions written below. No notes or texts should be used on the exam.

1 Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is continuous. If $s \geq 0$, let $I(f,s) = \displaystyle\int_{\partial\Delta(0,s)} f(z)\, dz$ (so $I(f,s)$ is just the line integral of f around the boundary of the circle of radius s centered at 0.). Prove: $\lim\limits_{s\to 1} I(f,s) = I(f,1)$.

2 a) An entire function f is of *exponential type* if $\exists A \geq 0$ and if $\exists C > 0$ so that $\forall z \in \mathbb{C}$ with $|z| > C$, $|f(z)| \leq Ce^{A|z|}$. Prove that the collection of entire functions of exponential type is closed under differentiation: that is, if f is of exponential type, so is f'. Indeed, show that if the constant A is valid for f, then the same constant is valid for f' (with a possibly different C).

b) Show that the analogous statement for C^∞ functions on $\mathbb{R}$ is false. That is, a C^∞ function g on $\mathbb{R}$ is of *exponential type* if $\exists A \geq 0$ and if $\exists C > 0$ so that $\forall x \in \mathbb{R}$ with $|x| > C$, $|g(x)| \leq Ce^{A|x|}$. Give an example of a function satisfying this condition for some A whose first derivative does **not** satisfy this condition for *any* A.

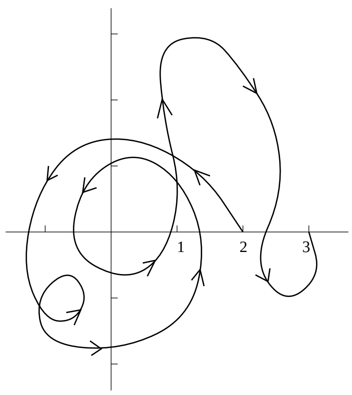
3 a) Suppose f is entire and that f is *doubly-periodic* with periods 1 and i: that is, $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Prove that f is constant.

b) Guess necessary and sufficient conditions on the periods of a doubly-periodic entire function which would guarantee that all such functions are constant. Comment on the proof of your conjecture.

4 The piecewise differentiable curve C is shown in the picture to the right. Explain your answers to the following questions as well as you can.

a) If we start with the "real" or "principal" value of $\log 2$ on this curve, and follow log continuously along the curve, what value do we get for $\log 3$?

b) What is $\displaystyle\int_C z^2 + 3z\, dz$?



The curve C

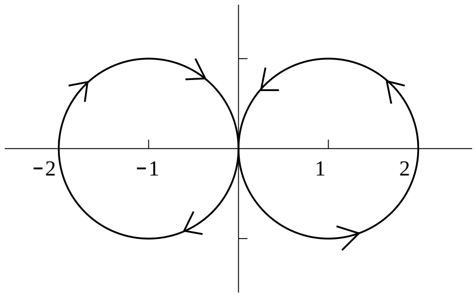
OVER

5 The piecewise differentiable closed curve Γ shown in the accompanying picture is $\partial\Delta(1,1) - \partial\Delta(-1,1)$, the function $h(z) = \dfrac{e^z}{(z-a)(z-b)}$, and $J(a,b) = \displaystyle\int_\Gamma h(z)\,dz$.

a) Suppose a and b are both in $\Delta(1,1)$ and $a \neq b$. What is $\displaystyle\lim_{a\,\&\,b\to 0} J(a,b)$?

b) Suppose $a \notin \overline{\Delta}(-1,1) \cup \overline{\Delta}(1,1)$ and $b \in \Delta(1,1)$. What is $\displaystyle\lim_{a\,\&\,b\to 0} J(a,b)$?

c) Suppose $a \in \Delta(-1,1)$ and $b \in \Delta(1,1)$. What is $\displaystyle\lim_{a\,\&\,b\to 0} J(a,b)$?



The curve Γ