October 24, 1997

Math 503 midterm exam

Please take two hours for this exam, use no book or notes, and explain your answers.

1 Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is continuous. If $s \geq 0$, let $I(f,s) = \int_{\partial\Delta(0,s)} f(z)\, dz$ (so $I(f,s)$ is just the line integral of f around the boundary of the circle of radius s centered at 0.). Prove: $\lim_{s\to 1} I(f,s) = I(f,1)$.

2 a) An entire function f is of *exponential-polynomial type* if $\exists A \geq 0$ and if $\exists C > 0$ so that $\forall z \in \mathbb{C}$ with $|z| > C$, $|f(z)| \leq Ce^{(C|z|^A)}$. Prove that the collection of entire functions of exponential-polynomial type is closed under differentiation: that is, if f is of exponential-polynomial type, so is f'. Indeed, show that if the constant A is valid for f, then the same constant is valid for f' (with a possibly different C).

b) Show that the analogous statement for C^∞ functions on $\mathbb{R}$ is false. That is, a C^∞ function g on $\mathbb{R}$ is of *exponential-polynomial type* if $\exists A \geq 0$ and if $\exists C > 0$ so that $\forall x \in \mathbb{R}$ with $|x| > C$, $|g(x)| \leq Ce^{(C|x|^A)}$. Give an example of a function satisfying this condition for some A whose first derivative does **not** satisfy this condition for *any* A.

3 Show that if $0 < r < R$ then there exists an $\epsilon > 0$ such that

$$\sup\{|p(z) - z^{-1}| : r < |z| < R\} > \epsilon,$$

for every polynomial $p(z)$.

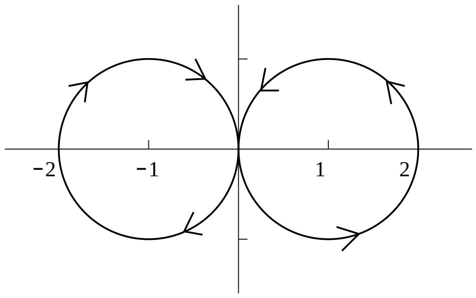
4 An entire function f with $f(0) = 0$ has the property that $\mathrm{Re}\, f(z) \to 0$ as $|z| \to \infty$. Show that $f(z) = 0$ for every z in $\mathbb{C}$.

5 The piecewise differentiable closed curve Γ shown in the accompanying picture is $\partial\Delta(1,1) - \partial\Delta(-1,1)$, the function $h(z) = \dfrac{e^z}{(z-a)(z-b)}$, and $J(a,b) = \int_\Gamma h(z)\, dz$.

a) Suppose a and b are both in $\Delta(1,1)$ and $a \neq b$. What is $\lim_{a\ \&\ b\to 0} J(a,b)$?

b) Suppose $a \notin \overline{\Delta}(-1,1) \cup \overline{\Delta}(1,1)$ and $b \in \Delta(1,1)$. What is $\lim_{a\ \&\ b\to 0} J(a,b)$?

c) Suppose $a \in \Delta(-1,1)$ and $b \in \Delta(1,1)$. What is $\lim_{a\ \&\ b\to 0} J(a,b)$?



The curve Γ