December 17, 1997

Math 503 final exam

Please take two hours for this exam, use no book or notes, and explain your answers.

1 Let C be a simple closed curve enclosing the points $0, 1, 2, \ldots, k$ in $\mathbb{C}$. Evaluate the integrals

$$I_k = \int_C \frac{dz}{z(z-1)\cdots(z-k)}\ , \quad k = 0, 1, \ldots$$

$$J_k = \int_C \frac{(z-1)\cdots(z-k)}{z}\, dz\ , \quad k = 0, 1, \ldots$$

Note: the product on the top of the integrand for J_k is 1 when $k = 0$.

2 Prove or disprove: if f is an entire function which maps every unbounded sequence to an unbounded sequence, then f is a polynomial.

3 Prove that the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$ and the punctured unit disc $\triangle^* = \triangle(0,1)\backslash\{0\}$ are *not* biholomorphic.

4 Find R large enough so that all of the roots of $p(z) = 2z^5 + (1+i)z^3 - 7z^2 + 9$ are inside the disc $\triangle(0, R)$. Prove your assertion.

5 Suppose that U is a connected open subset of $\mathbb{C}$, and that S is a discrete subset of U. Prove that $U\backslash S$ is also a connected open subset of $\mathbb{C}$.

6 Prove that a positive harmonic function on $\mathbb{R}^2$ is constant.