February 29, 2000

Math 504: Complex Variables (Spring, 2000)

F1 Create an entire function whose zero set (with multiplicities) is $\{(n,n)\}_{n\in\mathbb{N}}$. That is, try to get an explicit formula for a function which has a zero of order n at every positive integer n. (One way would be to use Weierstrass factors. Please provide proof that your factors behave the way they should, explicitly, in this case. Information about these factors should be in any standard complex variables text.)

F2 Find the Bergman kernel for the upper half plane H (those z's with positive imaginary part).

F3 Find a complete orthonormal set for the holomorphic functions which are square integrable in an annulus.

Note Hille, in *Analytic Function Theory*, volume II, mentions that an explicit formula for the Bergman kernel in an annulus can only be written using the Weierstrass $\wp$ function.

F4 Show that the unit ball in the Bergman space is normal. That is, any sequence of square-integrable holomorphic functions with L^2 norms bounded by 1 must have a subsequence which converges u.c.c.

F5 When are polynomials dense in $\mathcal{O}(U)$? When are polynomials dense in $\mathcal{B}(U)$?

The following problems are from the text *Complex Variables: An introduction* by Carlos Berenstein and Roger Gay.

F6 Suppose U is an open subset of $\mathbb{C}$. Prove that the collection of functions in $\mathcal{O}(U)$ which have U as domain of holomorphy is dense in $\mathcal{O}(U)$.

Hint If U is a domain of holomorphy for f, and $g\in\mathcal{O}(U)$, consider $g+tf$ when $0<t$.

F7 Let Ω be a simply connected open set in $\mathbb{C}$, f, $g\in\mathcal{O}(\Omega)$. What are the necessary and sufficient conditions on f and g in order that the equation $X^2+fX+g=0$ admits a solution $X\in\mathcal{M}(\Omega)$?

F8 Show that if $\{a_n\}_{n\in\mathbb{Z}}$ satisfies the condition $\sum_n |a_n|^2<\infty$, then the function $g(z)=\displaystyle\sum_n (-1)^n a_n \frac{\sin z}{z-n\pi}$ is entire, takes the values $g(n\pi)=a_n$ and $\displaystyle\int_{-\infty}^{\infty}|g(x)|^2\,dx<\infty$.

F9 Find all solutions $f\in C^\infty(\mathbb{C})$ of the equation $\dfrac{\partial f}{\partial\overline{z}}=g$ for: (a) $g(z)=z$; (b) $g(z)=\overline{z}$; (c) $g(z)=z^m\overline{z}^n$, $m,n\in\mathbb{N}$; (d) $g(z)=e^{\overline{z}}$; (e) $g\in\mathcal{O}(\mathbb{C})$.

F10 Suppose $E=D_1(0)$. We say that a function f belongs to $C^\infty\left(\overline{E}\right)$ if f is the restriction to $\overline{E}$ of a function which is defined and C^∞ in a neighborhood of $\overline{E}$. Let $g(z)=\exp\left(-\dfrac{1}{1-|z|^2}\right)$ for $z\in E$, $g|_{\partial E}=0$. Show that there is $f\in C^\infty\left(\overline{E}\right)$ such that $\dfrac{\partial f}{\partial\overline{z}}=g$ in $\overline{E}$.