Math 504: Complex Variables (Spring, 2000)

In our homework problems, unless otherwise specified a harmonic function is a C^2 real-valued function u satisfying $\triangle u = 0$ in its domain.

The Bergman idea is important and depends on problem **C1** proved by Mr. Wilckens. The first three problems ask you to see if the analogs of **C1** are true in some other settings. Problem **G1** is definitely part of the course. The next two problems are for the adventurous or for those who are more interested in analysis.

G1 Suppose $u \in C^2\left(D_{1+\epsilon}(0)\right)$ for some $\epsilon > 0$ and that $\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0^*$ always. Is the inequality $|u(0)| \le T\|u\|_{L^2(D_1(0))}$ true for some $T > 0$ not depending on u?

G2 Suppose $u \in C^2\left(D_{1+\epsilon}(0)\right)$ for some $\epsilon > 0$ and that $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0^{**}$ always. Is the inequality $|u(0)| \le T\|u\|_{L^2(D_1(0))}$ true for some $T > 0$ not depending on u?

G3 Suppose $u \in C^2\left(D_{1+\epsilon}(0)\right)$ for some $\epsilon > 0$ and that $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0^{***}$ always. Is the inequality $|u(0)| \le T\|u\|_{L^2(D_1(0))}$ true for some $T > 0$ not depending on u?

G4 Show that the set of positive harmonic functions on a connected open set is equicontinuous.

G5 Show that the zero set of a nonconstant entire harmonic function is nonempty and unbounded.

G6 Why is the Dirichlet problem usually stated for a *bounded* open set? Give an example of an unbounded open set so that there an infinite dimensional set of functions harmonic on the set which can be extended to be continuous on the closure of the set, all with zero boundary values.

G7 A random real polynomial in x and y is hardly ever harmonic. Prove this statement after making it precise in *some* sense.

Suggestions Perhaps consider all polynomials of degree $\le n$. What is the dimension of that vector space? What about the harmonic polynomials? What happens as $n \to \infty$? Or make the problem discrete: select the degree and coefficients of a polynomial with integer coefficients "at random" (you should specify what that means). Show that such a polynomial will rarely be harmonic.

G8 a) Show that an increasing, bounded sequence of real-valued continuous functions on $[a, b] \subset \mathbb{R}$ which converges pointwise need not converge uniformly.

b) Show that an increasing, bounded sequence of real-valued continuous functions on $[a, b] \subset \mathbb{R}$ which converges pointwise to a continuous function must converge uniformly.

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*An *elliptic* equation. **The wave equation, a *hyperbolic* equation. ***The heat equation, a *parabolic* equation.