March 22, 2000

Math 504: Complex Variables (Spring, 2000)

H1 Suppose $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of harmonic functions in a domain U. If $u_j \to u$ uniformly on compact subsets of U, show that $\frac{\partial u}{\partial x}$ exists, and that $\frac{\partial u_j}{\partial x} \to \frac{\partial u}{\partial x}$ uniformly on compact subsets of U.

Note Of course one can apply this result repeatedly to get any derivative of the sequence behaving similarly. The result will follow easily if you prove it first when U is a disc. Use the Poisson integral *inside* the disc.

H2 Suppose u is harmonic in $D_1(0)$. Show that u has a unique harmonic conjugate $v = C(u)$ characterized by $v(0) = 0$. Prove that the function $u \mapsto C(u) = v$ is linear and continuous with the u.c.c. topology. (You may use the results of the previous problem.)

H3 a) For $n \in \mathbb{N}$ find an explicit holomorphic function f_n with the following properties:

$f_n\colon D_1(0) \to S = \{|\mathrm{Re}\, z| < 1/n\}$ is biholomorphic; $f_n\left(D_1(0) \cap \mathbb{R}\right) = S \cap \mathbb{R}$; $f_n(0) = 0$.

b) Let $f_n = u_n + iv_n$. Surely $u_n \to 0$ uniformly in $D_1(0)$ but $\sup_{z\in D_1(0)} v_n(z) = \infty$ for all n. Doesn't this contradict the final conclusion of the previous problem?

The following problem will be used in class.

H4 a) Write $\triangle$ in polar coordinates. Conclude that any rotationally symmetric harmonic function must be of the form $A\log r + B$ where $r = |z|$.

b) Suppose u is harmonic in $D_1(0) \setminus \{0\}$. Define $U(r) = \int_0^{2\pi} u(re^{i\theta})\, d\theta$. Prove that U is also harmonic.

We know: for K compact contained in U open in $\mathbb{C}$, there's a smallest positive $\mathbf{H}_{K,U} \geq 1$ so that $\frac{h(x)}{h(y)} \leq \mathbf{H}_{K,U}$ for all $x, y \in K$ and all functions positive h harmonic in U.

H5 If U is a disc, show that $\mathbf{H}_{K,U} - 1$ and the diameter of K approach 0 together.

H6 Show that there are K's whose diameters $\to 0$ with $\mathbf{H}_{K,U} \to \infty$.

The following notorious problem is from the text *Banach Spaces of Analytic Functions* by Kenneth Hoffman. Hoffman's "analytic" is our "holomorphic".

H7 Let f be an analytic function in the unit disc *without zeros* satisfying $|f| \leq 1$. Prove that $\sup_{|z|\leq 1/5} |f(z)|^2 \leq \inf_{|z|\leq 1/7} |f(z)|$.

H8 In this problem please allow harmonic functions to be complex-valued. Since $\triangle$ is a real differential operator, this is equivalent to asking that the real and imaginary parts of the function be harmonic (but not necessarily related in any other way). The factorization of $\triangle$ suggested in problem **D10** may be useful.

a) If f is harmonic and $zf(z)$ is harmonic, then f is analytic. (This problem is also from Hoffman's book.)

b) Is the statement still true when the function z is replaced by any complex analytic function, $q(z)$? That is, given $q(z)$ complex analytic, is the following correct?

If f is harmonic and $q(z)f(z)$ is harmonic, then f is analytic.