Math 504: Complex Variables (Spring, 2000)

D will denote the open unit disc in $\mathbb{C}$ in the problems below. The following problem is from John Conway's *Functions of One Complex Variable II.*

I1 Let $p(z, \overline{z})$ be a polynomial in z and $\overline{z}$ and find a formula for the function u that is harmonic on D, continuous on cl D, and equal to $p(z, \overline{z})$ on ∂D.

The following problem is from the text *Introduction to Complex Analysis* by Rolf Nevanlinna and V. Paatero. The problem after it is almost from that book.

I2 Let the function $w(z) = u + iv$ be analytic in the half-plane $\operatorname{Im} z \geq 0$ (including the point ∞). Prove that $w(z) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} u(t) \frac{dt}{t-z} + iC$, where C is a real constant.

I3 If $u(x, y)$ is harmonic, is there always a non-constant C^2 function $v(x, y)$ so that the product $u(x, y)v(x, y)$ is harmonic?

The following three problems are from the text *Complex Analysis* by Lars Ahlfors.

I4 Show that the functions $|x|$, $|z|^\alpha (\alpha \geq 0)$, $\log(1 + |z|^2)$ are subharmonic.

I5 If v is continuous together with its partial derivatives up to the second order, prove that v is subharmonic if and only if $\triangle v \geq 0$. *Hint:* For the sufficiency, prove first that $v + \epsilon x^2$, $\epsilon > 0$, is subharmonic. For the necessity, show that if $\triangle v < 0$, the mean value over a circle would be a decreasing function of the radius.

I6 If $\Omega = D_1(0) \setminus \{0\}$ and if f is given by $f(\zeta) = 0$ for $|\zeta| = 1$, $f(0) = 1$, show that all functions $v \in \mathcal{P}_f$ are ≤ 0 in Ω.

And this problem is from Forster's *Lectures on Riemann Surfaces.*

I7 Suppose $Y \subset \mathbb{C}$ is open, $a \in \partial Y$ and there exists a line segment $S = \{\lambda a + (1 - \lambda)b : 0 \leq \lambda \leq 1\}$ with $b \neq a$ so that $Y \cap S = \emptyset$. Show that a is a regular boundary point of Y.

I8 Suppose H is defined on ∂D by requiring that H be 0 in the lower half plane and 1 in the upper half plane. Find the solution to the Dirichlet problem for H. If h is the solution, sketch the level set of h for $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.
Comment This is easily done by conformal mapping. What is the boundary behavior of h? Where *must* its limiting values be equal to H by general theory (and why)? What happens elsewhere?

I9 Let $A_{r,R}$ be the annular region defined by the restriction $r < |z| < R$. Let $t_{r,R}$ be the boundary "data" which is 1 on the inner circle, and 0 on on the outer circle, and let $s_{r,R}$ be the solution to the indicated Dirichlet problem. What happens to $s_{r,1}$ as $r \to 0^+$? What happens as $r \to 1^-$?

I10 Let S be the unit square ($x + iy$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$). Define boundary information Q by asking that Q be 0 on the bottom (where $x = 0$) and the sides (where $y = 0$ and $y = 1$). Let's suppose that Q on the top (the last side) is either $2x(1 - x)$ or $|1 - 2x|$ (you choose one of them!). Q is continuous on ∂S and there is a solution, q to the Dirichlet problem. What is q (try to give as explicit a formula as possible) and what is its boundary behavior? What is $q(.5, .5) \pm .001$? Sketch the level sets where q is $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

Comment Try separation of variables, Fourier series, etc. Look at sums of $\left\{ \begin{array}{l} \sin(nx) \\ \cos(nx) \end{array} \right.$ multiplied by $\left\{ \begin{array}{l} \cosh(ny) \\ \sinh(ny) \end{array} \right.$. I suspect you will need computer help for the last two questions: only approximate answers are requested.