January 17, 2000

Math 504: Complex Variables (Spring, 2000)

Let L denote the subset of $\mathbb{R}^2$ defined by $\{x = 0 \;\&\; y \geq 0\} \cup \{x \geq 0 \;\&\; y = 0\}$ (the union of the non-negative x- and y-axes).

A1 Prove that L is the image of a bijective C^∞ map of $\mathbb{R}$ to $\mathbb{R}^2$. That is, there are two C^∞ functions $a(t)$ and $b(t)$ defined on all of $\mathbb{R}$ so that the mapping $t \to (a(t), b(t))$ is 1-to-1 and onto L. Also, prove that L is not the image of a regular C^∞ map. Here "regular" means that for all $t \in \mathbb{R}$, either $a'(t)$ or $b'(t)$ is non-zero. You likely will need to use the Implicit/Inverse Function Theorem.

A2 Prove that L is not the image of a real analytic map. That is, there does not exist a pair of real analytic functions $a(t)$ and $b(t)$ so that the image of $\mathbb{R}$ in $\mathbb{R}^2$ via the mapping $t \to (a(t), b(t))$ is L.* Note that although the real and imaginary parts of a holomorphic function are always real analytic, it is false that any pair of real analytic functions can be assembled to create a holomorphic function. Example: $x - iy$.

A3 Prove that $\mathcal{A}(\mathbb{R})$, the set of real analytic functions defined on all of $\mathbb{R}$, is a ring using the usual pointwise addition and multiplication of functions as ring operations.*

A4 Prove that any positive element of $\mathcal{A}(\mathbb{R})$ has a cube root in $\mathcal{A}(\mathbb{R})$. Is this statement still true if the word "positive" is replaced by non-negative or just deleted? If you decide the statement would then be false, verify your assertion with example(s). Is there any additional hypothesis which would give the desired conclusion?*

A5 If U is open in $\mathbb{C}$, let $\mathcal{O}(U)$ be the collection of holomorphic (complex analytic) functions in U. Given $f \in \mathcal{A}(\mathbb{R})$, prove that there is a connected open set $U \subset \mathbb{C}$ with $\mathbb{R} \subset U$ and an $F \in \mathcal{O}(U)$ so that $F|_{\mathbb{R}} = f$. Can the same U be used for all f's?

A6 Prove that the U in the previous problem can generally not be chosen to be a uniform neighborhood of $\mathbb{R}$. That is, given $\epsilon > 0$, define V_ϵ to be those complex numbers z with $|\operatorname{Im} z| < \epsilon$. Verify by giving an example as explicitly as possible that there exists $f \in \mathcal{A}(\mathbb{R})$ such that for all $\epsilon > 0$, $f \neq F_\epsilon|_{\mathbb{R}}$ for any $F_\epsilon \in \mathcal{O}(V_\epsilon)$.

A7 This problem is exactly quoted from the text *Analytic Functions*, written by Stanislaw Saks and Antoni Zygmund and published in 1938. $M(r; F)$ is $\max\limits_{|z|=r} |F(z)|$.

> Considering an everywhere convergent power series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^{\lambda_n}$, where $\{\lambda_n\}$ is a sufficiently rapidly increasing sequence of natural numbers, show that for an arbitrary real function $\varphi(r)$ defined for $r \geq 0$, bounded in every finite interval and tending to infinity together with r, there exists an entire function $F(z)$ such that $M(r; F) \geq \varphi(r)$ for $r \geq 0$. In other words, the function $M(r; F)$ can grow arbitrarily rapidly (Poincaré).

A8 Prove that any closed subset of $\mathbb{R}$ is the zero set of a non-negative C^∞ function.

A9 Suppose that $f \in C^0(\mathbb{R})$, and that $E \in C^0(\mathbb{R})$ is always positive. Prove that there is $g \in C^\infty(\mathbb{R})$ with $|f(x) - g(x)| < E(x)$ for all $x \in \mathbb{R}$.

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* Results from complex analysis may substantially simplify the solution of this problem.