Fun with Distributions!

April 10, 2000

# Math 504: Complex Variables (Spring, 2000)

**J1** Find a distribution solution of $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ (the homogeneous wave equation) defined in $\mathbb{R}^2$ which isn't a continuous function in $\mathbb{R}^2$. More precisely, show that there is no function $w$ continuous in $\mathbb{R}^2$ with $w = u$ almost everywhere. Verify your claims, please!

**J2** a) Every function in $L^1_{\mathrm{loc}}(\mathbb{C})$ is associated to a distribution.
b) $\log r$ is in $L^1_{\mathrm{loc}}(\mathbb{C})$.
c) What is $\triangle(\log r)$?

**Comment** The result of this problem is *very* important. Physicists have believed it for a long time. One proof of the answer to c) uses Green's identities and words like "charge" and "potential". Now mathematicians believe this result too.

**J3** a) Define $T\colon \mathcal{O}(\mathbb{C}) \to \mathbb{C}$ by the formula $T(f) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}$. Prove that $T(f)$ is well-defined, and that $T$ is continuous with respect to the u.c.c. topology on $\mathcal{O}(\mathbb{C})$.
b) $T$ is *not* well-defined on $\mathcal{D}(\mathbb{C})$.

**Comment** Distributions aren't enough for all aspects of complex analysis!

The next two problems are versions of some in Forster's *Lectures on Riemann Surfaces*.

**J4** Suppose $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of distributions defined in an open subset $U$ of $\mathbb{C}$. Define "$T_j \to T$" (where $T$ is also a distribution) to mean $T_j(f) \to T(f)$ for every $f \in \mathcal{D}(U)$. Prove that if $T_j \to T$, then $\frac{\partial T_j}{\partial x} \to \frac{\partial T}{\partial x}$.

**J5** A sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ defined in an open subset $U$ of $\mathbb{C}$ *converges weakly* to a continuous function $f$ if $\int_{\mathbb{C}} f_j \phi \, d\mathcal{L} \to \int_{\mathbb{C}} f \phi \, d\mathcal{L}$ for all $\phi \in \mathcal{D}(U)$. Here $\mathcal{L}$ is two-dimensional Lebesgue measure. Prove that if every $f_j$ is harmonic (resp. holomorphic) then $f$ must be harmonic (resp. holomorphic).

The next three problems are from J. E. Marsden's text *Elementary Classical Analysis*. $\delta$ is the Dirac $\delta$-function at 0.

**J6** Show that $\delta''(f) = f''(0)$ for $f \in \mathcal{D}(\mathbb{R})$.

**J7** Prove that $\sqrt{\frac{n}{\pi}}\, e^{-nx^2} \to \delta$ as distributions on $\mathbb{R}$.

**J8** Find a sequence of continuous functions $\{g_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}$ so that $g_n \to \delta'$ as distributions.

**J9** Suppose $f$ is a periodic $L^1_{\mathrm{loc}}$ function on $\mathbb{R}$ with period $2\pi$. If $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\, dt$, show that $f = \sum_{-\infty}^{\infty} a_n e^{int}$ as distributions.

**J10** Suppose that $\{a_n\}_{n\in Z}$ is a sequence of complex numbers with *polynomial growth*: there are $K \in \mathbb{N}$ and $c \in \mathbb{R}$ so that $|a_n| \le c|n|^K$ for all $n \in Z$. Then $\sum_{-\infty}^{\infty} a_n e^{inx}$ converges to a distribution defined on $\mathbb{R}$.

**J11** "Let $\mathsf{L}$ denote the subset of $\mathbb{R}^2$ defined by $\{x = 0 \;\&\; y \ge 0\} \cup \{x \ge 0 \;\&\; y = 0\}$ (the union of the non-negative $x$- and $y$-axes)."* Let $S$ be the distribution defined by integrating over $\mathsf{L}$**. What is $\frac{\partial S}{\partial z}$? Can you find $T$ so that $\frac{\partial S}{\partial z} = T$?

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* Of course this is quoted from the beginning of homework **A**! We end as we began.

** Use $dx$ and $dy$ appropriately.