January 31, 2000

Math 504: Complex Variables (Spring, 2000)

In this problem set, U will be a fixed open subset of $\mathbb{C}$. "L^2" in the first two problems below is with respect to area measure in the plane.

C1 a) Suppose that $f \in L^2(D_\varepsilon(0)) \cap \mathcal{O}(D_\varepsilon(0))$ for some $\varepsilon > 0$. Express the L^2 norm of f in terms of the power series coefficients of f, and prove that, given $j \in \mathbb{N} \cup \{0\}$, there is $C = C(j, \varepsilon) > 0$ so that $|f^{(j)}(0)| \leq C\|f\|_{L^2(D_\varepsilon(0))}$.

b) Suppose K is compact in U, and $\varepsilon > 0$ is such that $K_\varepsilon \subset U$. Given $j \in \mathbb{N} \cup \{0\}$, prove that there is $C = C(j, \varepsilon) > 0$ so that $\|f^{(j)}\|_K \leq C\|f\|_{L^2(K_\varepsilon)}$ for all $f \in \mathcal{O}(U)$.

C2 Please assume the results of the previous problem.

a) Suppose $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of functions in $L^2(U) \cap \mathcal{O}(U)$. Prove that if $\{f_k\}_{k\in\mathbb{N}}$ is L^2-Cauchy, then $\{f_k\}_{k\in\mathbb{N}}$ is u.c.c.-Cauchy.

b) If f is the u.c.c. limit of the sequence in a), then f is also the L^2 limit of the sequence.

C3 Suppose $\{f_j\}_{j\in\mathbb{N}}$ is a sequence of functions in $\mathcal{O}(U)$ which converges *pointwise* to f in U. Prove that there is a dense open subset V of U so that $f \in \mathcal{O}(V)$ and $f_j \to f$ u.c.c. in V. What happens if the sequence of f_j's has pointwise bounded modulus at every $z \in U$?

C4 There is $f \in \mathcal{M}(\mathbb{C})$ which has a pole of order j at each $j \in \mathbb{N}$ (and those are its only poles). Exhibit such a function precisely, and give $a, b \in \mathbb{Q}$ so that $|f(1+i)-(a+bi)| < .001$.

C5 There is $f \in \mathcal{M}(\mathbb{C})$ which has simple poles at 2^j for each $j \in \mathbb{N}$ (and those are its only poles), and whose principal part at 2^j is $\dfrac{3^j}{z-2^j}$ for each j. Exhibit such a function precisely, and give $a, b \in \mathbb{Q}$ so that $|f(1+i) - (a+bi)| < .001$.

C6 (Ridiculous polynomials, **I**) Show that there is a sequence of polynomials $P_n(z) \in \mathbb{C}[z]$ so that as $n \to \infty$, $P_n(0) = 1$ and $P_n(z) \to 0$ for $z \neq 0$.

C7 (Ridiculous polynomials, **II**) Show that there is a sequence of polynomials $P_n(z) \in \mathbb{C}[z]$ so that as $n \to \infty$, $P_n(z) \to 1$ for $\mathrm{Re}\, z \geq 0$ and $P_n(z) \to 0$ for $\mathrm{Im}\, z < 0$.

C8 (Ridiculous polynomials, **III**) Show that there is a sequence of polynomials $P_n(z) \in \mathbb{C}[z]$ so that as $n \to \infty$, $P_n(z) \to 1$ if $|z| = 1$ and $P_n(z) \to 0$ otherwise.

C9 (Ridiculous polynomials, **IV**) Show that there is a sequence of polynomials $P_n(z) \in \mathbb{C}[z]$ so that as $n \to \infty$, $P_n(z) \to 0$ for all $z \in \mathbb{C}$ <u>but</u> $P_n \not\to 0$ uniformly on all compact subsets of $\mathbb{C}$.

C10 (Ridiculous polynomials, **V**) Prove that there is $f \in \mathcal{O}(D_1(0))$ so that for all $\zeta \in \partial D_1(0)$ and all $\varepsilon > 0$, the closure of $\{f(s\zeta) : 1-\varepsilon < s < 1\}$ is all of $\mathbb{C}$. (For such f's, radial limits are wildly non-existent! One way to show that such a function exists is to write it an the u.c.c. limit in $D_1(0)$ of a sequence of "ridiculous" polynomials.)

C11 Give some metric d on $\mathcal{O}(\mathbb{C})$ where convergence in d is equivalent to u.c.c. Find positive numbers A and B with $A \leq d(z, z^2) \leq B$. If the diameter of $\mathcal{O}(\mathbb{C})$ as measured by d is finite, please supply a valid B less than the diameter (include verification of this).