February 9, 2000

Math 504: Complex Variables (Spring, 2000)

In this problem set, U is a fixed open *connected* subset of $\mathbb{C}$. $Z(f)$ will denote the set $f^{-1}(0)$, the set of zeros of f.

D1 $\mathcal{O}(U)$ is a ring. Is $\mathcal{O}(U)$ a PID?

D2 $\mathcal{O}(U)$ is a ring. Is $\mathcal{O}(U)$ a UFD?

D3 $\mathcal{O}(U)$ is a ring. Is $\mathcal{O}(U)$ Noetherian?

D4 $\mathcal{O}(U)$ is a ring. Is $\mathcal{O}(U)$ Artinian?

D5 Suppose $f, g \in \mathcal{O}(U)$ and $Z(f) \cap Z(g) = \emptyset$. Prove that there are $u, v \in \mathcal{O}(U)$ so that $1 = uf + vg$. Extend this statement to finite sums.

D6 Suppose $\{f_n\}_{n\in\mathcal{N}}$ is a sequence of functions in $\mathcal{O}(U)$ with no zeros in common: $\bigcap_{n=1}^{\infty} Z(f_n) = \emptyset$. Is there a sequence of functions $\{g_n\}_{n\in\mathcal{N}}$ in $\mathcal{O}(U)$ so $\displaystyle\sum_{n=1}^{\infty} f_n(z)g_n(z) = 1$ for all $z \in U$?

Example Suppose $h(z) = \frac{\sin z}{z}$, an entire function with $h(0) = 1$. Let $j_n(z) = h(z-n)$ for $n \in \mathbb{Z}$ (a double-ended sequence). Are there entire functions $\{k_n\}_{n\in\mathbb{Z}}$ with $\displaystyle\sum_{n=-\infty}^{\infty} j_n(z)k_n(z)$ $= 1$? Of course the zero sets of any finite number of the j_n's have non-empty intersection, but *all* of these zero sets have no element in common.

D7 This problem is from the text *Classical Topics in Complex Function Theory* by Reinhold Remmert (English translation published in 1998).

If $g \in \mathcal{M}(U)$, show that there is $f \in \mathcal{O}(U)$ so that $f(z) \neq g(z)$ for all $z \in U$.

Hint: Start with a representation $g = \dfrac{f_1}{f_2}$ where f_1 and f_2 have no common zeros.

Comment on algebra (D8 & D9) The ring $\mathcal{O}(U)$ has *many* maximal ideals. Some of them are difficult to understand*, but the <u>closed</u> maximal ideals consist of those functions vanishing at a point of U. Here "closed" refers to the topology on $\mathcal{O}(U)$ given by u.c.c. convergence. This is not very difficult to verify, but is perhaps more difficult than a homework problem should be. Therefore proving it would count as *two* homework problems!

D10 If $g \in C^\infty(U)$, prove that there is $f \in C^\infty(U)$ with $\triangle f = g$. Here $\triangle = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

Hint Relate $\triangle$ and $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \bar{z}}$.

D11 Is there a Mittag-Leffler theorem for essential singularities? That is, surely there are functions with infinitely many isolated essential singularities (give an example!). If a collection of "principal parts" of such functions is given in, say, $\mathbb{C}$, can one "piece" them together to create a function holomorphic except for those singularities, with the given Laurent behavior at each singularity?

* Unless you're a logician!