Math 504: Complex Variables (Spring, 2000)

E1 Embed $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ in $\mathbb{C}^N$ for some N.

E2 Embed $\mathbb{C} \setminus \mathbb{Z}$ in $\mathbb{C}^N$ for some N.

E3 Suppose $\mathbb{C}_-$ is the subset of $\mathbb{C}$ without the non-negative reals: it contains only those z's either with $\operatorname{Im} z \neq 0$ or with $\operatorname{Im} z = 0$ and $\operatorname{Re} z > 0$. Embed $\mathbb{C}_-$ in $\mathbb{C}^N$ for some N.

E4 Suppose A is the annulus of z's in $\mathbb{C}$ with $1 < |z| < 2$. Embed A in $\mathbb{C}^N$ for some N.

Exponential-polynomial functions can be used to create formulas for certain functions verifying Weierstrass's Theorem. Any standard text discusses this. They are also important in the study of Fourier transforms and solutions of partial differential equations.

Definition An entire function f is of *exponential-polynomial type* if $\exists A \geq 0$ and if $\exists C > 0$ so that $\forall z \in \mathbb{C}$ with $|z| > C$, $|f(z)| \leq Ce^{\left(C|z|^A\right)}$.

E5 Prove that the collection of entire functions of exponential-polynomial type is a ring which is also closed under differentiation: that is, if f is of exponential-polynomial type, so is f'. Indeed, show that if the constant A is valid for f, then the same constant is valid for f' (with a possibly different C).
Hint Use the Cauchy integral formula for f' with radius $= 1$.

E6 Show that a statement similar to the previous one for real analytic functions on $\mathbb{R}$ is false. That is, a real analytic function g on $\mathbb{R}$ is of *exponential-polynomial type* if $\exists A \geq 0$ and if $\exists C > 0$ so that $\forall x \in \mathbb{R}$ with $|x| > C$, $|g(x)| \leq Ce^{(C|x|^A)}$. Give an example of a function satisfying this condition for some A whose first derivative does **not** satisfy this condition for *any* A.

E7 a) Show that polynomials are exponential-polynomial functions.
b) Verify that exponential-polynomial functions are *not* u.c.c. closed in $\mathcal{O}(\mathbb{C})$.

Definition For h defined on $\mathbb{R}$, the *Fourier transform*[1] of h is $\hat{h}(t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-itx} h(x)\, dx$.

E8 What is the Fourier transform of $\frac{1}{1+x^2}$?[2]

E9 What is the Fourier transform of $\frac{\sin x}{x}$?[3]

E10 a) Suppose that $h \in C^\infty(\mathbb{R})$, and the support of h is contained in $[-R, R]$ for some $R > 0$. Show that $\hat{h}$ is entire, and $\forall n \in \mathbb{N}$, $\exists C_n \geq 0$ so that $|\hat{h}(t)| \leq \dfrac{C_n e^{R|\operatorname{Im} t|}}{(1+|t|)^n}$.[4]

b) <u>After</u> you've done a), find and write a statement of the Paley-Wiener Theorem.

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[1] Maple computes Fourier transforms with a different normalization.

[2] Maple reports the answer is `Pi*(exp(-t)*Heaviside(t)+exp(t)*Heaviside(-t))`.

[3] Maple reports the answer is `-Pi*Heaviside(t-1)+Pi*Heaviside(t+1)`.

[4] Derivatives of $h \Longleftrightarrow$ multiplication of $\hat{h}$ by powers of it.