Lecture $14\,^1/_2$: Intermezzo

BIRTHDAY TIME!

Suppose you have a crowd of people. When will two of them share a birthday?*

Probabilistic answer

This question is studied in almost every elementary probability text. Suppose we had 3 people, and we want to figure out the chance that these 3 people all have distinct birthdays. The first person "selects" a birthday with no constraint. The second person has probability $\frac{364}{365}$ of selecting a birthday which misses the first person's. The third person has probability $\frac{363}{365}$ of selecting a birthday which misses both the first and second person's birthday. Thus the probability that 3 people have distinct birthdays (that everything described happens) is $\frac{(365)(364)(363)}{(365)(365)(365)}$. This is about .99180, and therefore a birthday will be shared with probability $1 - .99180 = .00820$.

Now suppose there are n people in the crowd. The probability that no two of them have the same birthday is

$$\overbrace{\left(\frac{365}{365}\right)\cdot\left(\frac{364}{365}\right)\cdot\left(\frac{363}{365}\right)\cdots\left(\frac{365-n+1}{365}\right)}^{n \text{ terms}}$$

and as n increases, the value of this fraction shrinks. For $n = 23$, it is about .49270, less than $\frac{1}{2}$. Therefore, "more than half the time", a crowd with at least 23 people will have some pair sharing birthdays.

This result seems hard to believe and has been called the *Birthday Paradox*: relatively few people (only 23) make it more likely than not that at least one of the 365 days in the year is at least "doubly occupied".

A more definite answer

If you have more than 365 people in the crowd, at least 2 of them will share a birthday.

* Please disregard leap years!