

## Lecture 15: Pigeons and patterns

### 15.1 Pigeonholes

The simplest way to show that patterns must appear uses *the pigeonhole principle*.

#### The Pigeonhole Principle

If there are more pigeons than pigeonholes, one pigeonhole must contain at least two pigeons.

The following application is based on contents of a webpage at the University of Manchester Institute of Science and Technology.

Some multiple of 12345 has a decimal expansion which can be written using only the digits 0 and 1.

Why is this true? Consider the 12346 numbers  $A_1, A_2, A_3, \dots, A_{12346}$  defined by the

formula  $A_n = \overbrace{111 \dots 111}^{n \text{ 1's}}$  (in decimal notation). Further define the  $R_n$ 's to be the remainders gotten from dividing the  $A_n$ 's by 12345. These integer remainders are scattered between 0 and 12344, a total of 12345 possibilities. These possibilities are the *pigeonholes*.

There are 12346 remainders: the *pigeons*. The pigeonhole principle implies that at least two of these  $R_n$ 's are equal. So we may suppose that there are remainders  $R_a$  and  $R_b$  where  $a > b$  so that  $R_a = R_b$ . Then  $A_a - A_b$  is divisible by 12345 because its remainder,

the difference  $R_a - R_b$ , is 0. What is  $A_a - A_b$ ? It must be  $\overbrace{111 \dots 111}^{(a-b) \text{ 1's}} \overbrace{000 \dots 000}^{b \text{ 0's}}$

It isn't too difficult to write a computer program which implements the scheme outlined above. For example, here's a multiple of 13 which is all 0's and 1's:  $85470 \cdot 13 = 1111110$ . But the number found may not be the smallest answering the question:  $77 \cdot 13 = 1001$  is also a solution which is quite a bit smaller! More critically, the process outlined above shows that certain numbers exist, but it doesn't find them. A direct search for the multiple of 12345 using the idea above could involve manipulation of numbers with 12345 digits, which is somewhat intimidating, even for a computer.

**Exercise** Find an explicit multiple of 12345 which has only 0's and 1's in its decimal expansion.

### 15.2 M&M's

Here's a question involving pattern inevitability taken from a webpage at Illinois State.

The colors of M&M's are red, green, brown, yellow, blue, and orange.

How many M&M's would you have to grab from a package to guarantee that you have grabbed at least three of one color?

Certainly a "random" three M&M's taken could have the same color. Let's analyze that probability. Here we use information supplied by Mars, Inc. of Hackettstown, New Jersey.

#### Brown

Fact: 30% of all M&M's are brown.

Consequence: if we choose three different M&M's independently, then  $(.3)^3 = .027$  of the time we'd have three brown M&M's.

### **Yellow & Red**

Facts: 20% of all M&M's are yellow; 20% of all M&M's are red.

Consequences: the chance of getting three yellow M&M's is  $(.2)^3 = .008$  and the chance of getting three red M&M's is the same.

### **Green & Orange & Blue**

Facts: 10% of all M&M's are green; the same is true for orange and for blue.

Consequences: The chance of getting three greens is  $(.1)^3 = .001$ , and this is the same as the chance of getting three oranges or three blues.

Let's assume that the manufacturer has supplied correct information and that the selection of the three M&M's is independent. Then the event of selecting three of the same color splits into six disjoint events, one for each color. The probability of selecting three of the same color is therefore  $(.3)^3 + 2(.2)^3 + 3(.1)^3 = .027 + 2(.008) + 3(.001) = .046$ , roughly one chance in twenty. This reasoning is, of course, probabilistic. Suppose we wanted to be certain of getting M&M's with the same color.

We could grab more. If we took 7 M&M's, then at least one color would appear twice. This is a consequence of the Pigeonhole Principle: the pigeonholes would be the color of the candy taken, and the pigeons would be each M&M. Taking 7 M&M's with 6 colors available is the same as having 7 "pigeons" (the 7 chosen M&M's) and 6 pigeonholes, one hole for each color chosen: two M&M's of any 7 must have the same color. Similar reasoning shows that any group of **13** M&M's must have three of the same color.

**Exercise** Carefully compute the probability of three M&M's of one color being grabbed in a random handful of  $N$ , where  $3 \leq N \leq 13$ . For example, the probability is .8 when  $N = 12$ . The probability will grow from .046 ( $N = 3$ ) to 1 ( $N = 13$ ).

The pigeonhole principle can produce mandatory "structure" in randomness.

**Current events** The distribution of M&M colors mentioned above is two years old. The *current* published values for color distribution are:

brown, 13%; yellow, 14%; red, 13%; blue, 24%; orange, 20%; green, 16%.

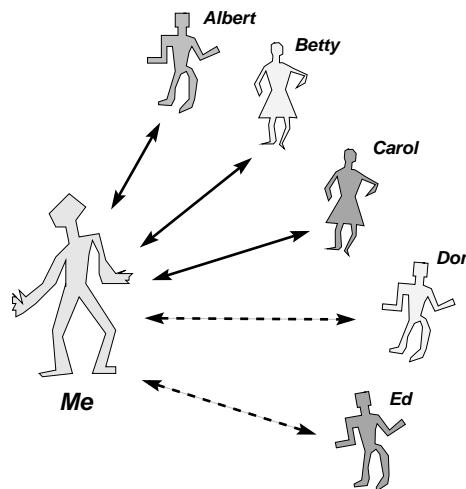
I don't know why the distribution has changed with time. Also, different M&M products have different color distributions. Clearly this is a complex phenomenon!

## Lecture 16: Friends, strangers, and coloring graphs

### 16.1 Mutual friends and mutual strangers

Let's suppose that any pair of people in the world are either mutual friends or mutual strangers. Take any six people in the world. Must there be a group of these people (more than two!) who are all mutual friends or mutual strangers?

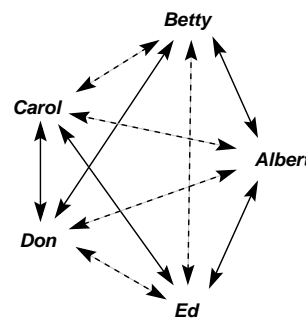
Suppose I (in the lovely picture: “*Me*”) am one of the people, and the others are *Albert*, *Betty*, *Carol*, *Don*, and *Ed*. Some of these five people are strangers to me, and some are my friends. By the pigeonhole principle, at least three of these must be one of these two alternatives (strangers to me or friends of mine). Let me assume that three are friends, and that *Albert* and *Betty* and *Carol* are these friends.



Consider *Albert* and *Betty* and *Carol* who could all be unacquainted with each other. So these people then would be mutual strangers. If this assumption is false, at least one pair of them, say *Albert* and *Betty*, would be friends. Then *Albert* and *Betty* and *Me* would be three mutual friends.

We could continue backtracking the assumptions that were made. What would happen if there were three people who were strangers to *Me*? Similar reasoning will again provide three mutual friends or three mutual strangers.

What if we considered a collection of five people? Look at the diagram displayed. The solid lines indicate that the people are mutual friends, while the dashed lines imply they are mutual strangers (better thought: “friends who haven't met”). Check carefully: each of the 10 relationships is shown. There is no triple of mutual friends or mutual strangers.



Here is the major conclusion:

*Any* group of six people must include a triple of mutual friends or a triple of mutual strangers but there can be groups of five people having no triple of mutual friends and no triple of mutual strangers.

### 16.2 Graphs and coloring four

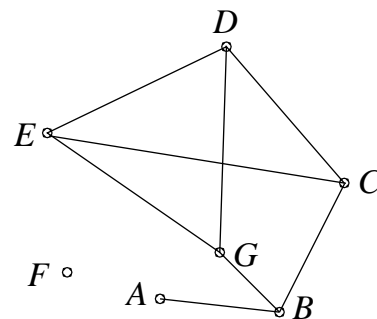
We just considered what could happen at a party with a group of people. We were interested in whether a small number of them were mutual friends or strangers. We are actually studying *graphs*\*.

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\* Mathematicians tend to overload simple words, making them serve several distinct meanings. The objects called graphs here are *not* the same as the graphs considered in analytic geometry, such as “the graph of  $y = x^2$ .” Sorry about this.

Here a graph will be a collection of vertices (the plural form of “vertex”) and edges between pairs of vertices. One graph is shown here. The vertex set of this graph has 7 elements,  $\{A, B, C, D, E, F, G\}$ ; the edge set has 8 elements,  $\{(A, B), (B, C), (B, G), (C, D), (C, E), (D, E), (D, G), (E, G)\}$ .

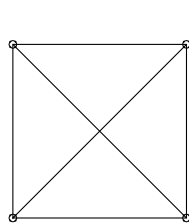
The vertex  $B$  is a member of three edges, and we say that the degree of  $B$  is 3:  $\deg(B) = 3$ . The degree of  $F$  is 0. The picture is a symbolic representation of the graph so the edges  $(C, E)$  and  $(D, G)$  don't really meet or intersect.



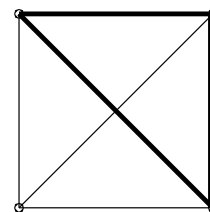
The concept of a graph has many variations (directed graph: edges with an orientation; weighted graph: edges with associated costs; a multigraph: more than one edge may connect vertices, etc.). There is also much specialized terminology.

Many simple questions having difficult answers have been asked about graphs. One of the most famous optimization problems asks about graphs. Suppose there is a list of cities and distances between them is given. Finding a tour visiting each of the cities with minimum travel distance is the Traveling Salesman Problem. This is certainly a difficult (NP!) problem. It has many applications\*.

Here we will look at complete graphs. A graph is *complete* if every pair of distinct vertices is connected by an edge. The complete graph on  $n$  vertices is called  $K_n$ .

 $K_4$ 

$K_4$  is shown at the left. It has 4 vertices and it has  $3 + 2 + 1 = 6$  edges. The picture at the right shows this sum: 3 counts the edges connecting a first vertex to all the other edges (edges drawn with standard thickness); 2 comes from connecting another vertex to all vertices except the first (edges drawn with increased thickness); 1 (the most heavily drawn edge) is from the next-to-last vertex, connecting it to the last vertex.

 $1 + 2 + 3 = 6$ 

More generally, the number of edges in  $K_n$  is  $\frac{n(n-1)}{2}$ . You can see this in several ways: by looking at the pairs of vertices ( $n$  times  $n - 1$ , since no vertex is connected to itself) and dividing by 2 (since the order of the pairs doesn't matter). Or you can count downwards and add:  $n + (n - 1) + (n - 2) + \dots + 2 + 1 = \frac{n(n-1)}{2}$ .

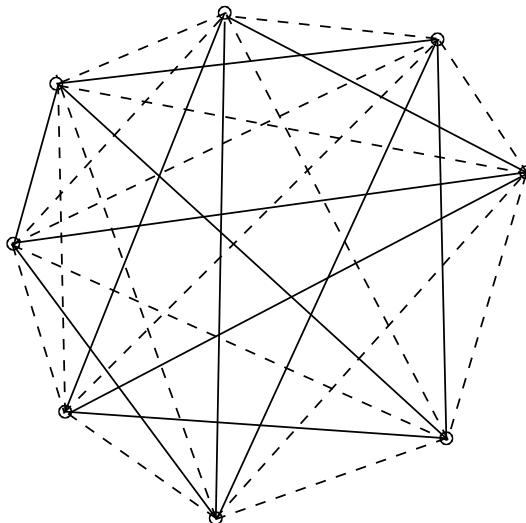
We will **color** the edges of  $K_n$ 's. Here I want to use only two colors, say **red** (indicated with ———) and **blue** (indicated with - - - -). Since each edge can be colored one of these **2** colors,  $K_n$  can be colored  $2^{(\# \text{ of decisions})} = 2^{n(n-1)/2}$  different ways. In the previous section, we actually verified the following statement:

If  $n \geq 6$  then every 2-colored  $K_n$  contains a monochromatic triangle ( $K_3$ ). If  $n \leq 5$ , there are colorings of  $K_n$ 's not containing a monochromatic triangle.

Is there a similar statement about  $K_4$ 's? That is, can we find a threshold number  $N$  so that there is some 2-colored  $K_{N-1}$  which contains *no* monochromatic  $K_4$  and so that every 2-colored  $K_N$  *must* contain at least one monochromatic  $K_4$ ? Statements of this type are not easy to check because the number of possible 2-colorings is very large.

\* See <http://www.tsp.gatech.edu/> for information, including lovely pictures.

For example, here's one 2-colored  $K_8$ . It has 28 edges and *no* monochromatic  $K_4$ . To check this claim, look at all of the possible  $K_4$ 's and see if they have one color. How many  $K_4$ 's are there? First find the number of distinct 4-tuples of vertices: you can choose the first vertex 8 ways, the second 7, then 6, and then 5. The result is 1680. But we get the same  $K_4$  if these 4 vertices are chosen in any order. The number of orders of 4 objects is  $4 \cdot 3 \cdot 2 \cdot 1 = 24$ . We need to check *only* 70 different  $K_4$ 's\*. You may know this is the binomial coefficient  $\binom{8}{4}$ .



Check all 70!

How **hard** is it to check for monochromatic  $K_4$ 's in a 2-colored  $K_N$ ? We might need to check the coloring of every  $K_4$  in the given  $K_N$ . There are  $\binom{N}{4}$  different  $K_4$ 's: that is, we just pick 4 vertices and look at the 6 edges connecting these vertices. The number of ways of choosing 4 objects from  $N$  objects is  $\binom{N}{4} = \frac{N(N-1)(N-2)(N-3)}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{N^4 - 6N^3 + 11N^2 - 6N}{24}$ . This is a polynomial of degree 4. We could search one 2-colored  $K_N$  for monochromatic  $K_4$ 's with a program having four “nested” loops. This is not really too bad. For example, when  $N = 15$ , there are 1,365  $K_4$ 's. But ... to check that *every* 2-colored  $K_N$  has a monochromatic  $K_4$ , we would need to check  $2^{N(N-1)/2}$  different 2-colored  $K_N$ 's. When  $N = 15$ , there are  $2^{105} = 40\,564\,819\,207\,303\,340\,847\,894\,502\,572\,032 \approx 4 \cdot 10^{31}$  different colorings to check. A direct check is not feasible even with huge computer resources (the return of P versus NP!). We have two tasks:

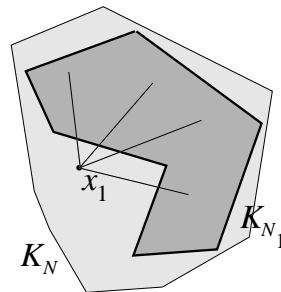
- I Find  $N$  large enough so that *every* 2-colored  $K_N$  has a monochromatic  $K_4$ . It isn't even clear that there is such an  $N$ , and therefore exhaustive searching may not succeed.
- II For suitable  $N$ 's, exhibit 2-colored  $K_N$ 's without any monochromatic  $K_4$ 's. Certainly  $N = 8$  works here (the example above!). But can larger  $N$ 's be used and still *not necessarily* have monochromatic  $K_4$ 's? Even here searching be impractical.

How can we find a monochromatic  $K_4$ ? Let's start with a very **BIG** 2-colored  $K_N$  and try to find a  $K_4$  with only one color. We search for a monochromatic  $K_4$  in a greedy way. First, pick *any* vertex: call it  $x_1$ . Look at the edges coming out of  $x_1$ : there are  $N - 1$  of them. Some are red and some are blue. There are at least  $\frac{1}{2}(N - 1)$  edges of one color, because if the number of edges of both colors were  $< \frac{1}{2}(N - 1)$ , then the total number of edges would be less than  $N - 1$ . Let's see: call that color  $C_1$ , and collect all the vertices which connect to  $x_1$  with edges of that color. Look at the complete graph (with assigned edge colorings!) on those other vertices. It is a  $K_{N_1}$  whose internal edges are colored red and/or blue. The “and/or” is used to emphasize that we have no control over the internal edges of  $K_{N_1}$ . Also,  $N_1 \geq \frac{1}{2}(N - 1)$ .

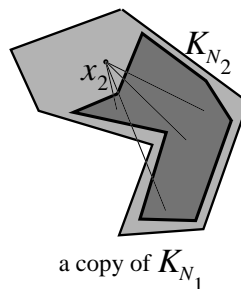
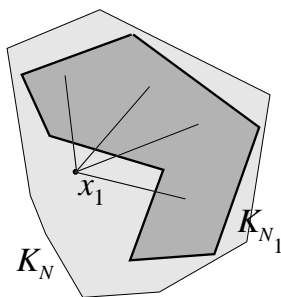
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\*  $70 = \frac{1680}{24}$ .

Here is an attempt to illustrate what has just been described. The lighter colored polygonal region “is”  $K_N$ . Then  $x_1$  is inside  $K_N$ , and is connected by edges of all one color to  $K_{N_1}$ , which takes up at least half of  $K_N$ . We don’t know whether the edges connecting the vertices of  $K_{N_1}$  to  $x_1$  are red or blue, only that they are all the same color.



Now pick a vertex  $x_2$  in  $K_{N_1}$ . Look at the edges coming out of  $x_2$  in  $K_{N_1}$ : there are  $N_1 - 1$  of them. There are at least  $\frac{1}{2}(N_1 - 1)$  edges of one color, because if the number of edges of both colors were  $< \frac{1}{2}(N_1 - 1)$ , then the total number of edges would be less than  $N_1 - 1$ . Call that color  $C_2$ . I am *not* asserting it is the same as  $C_1$ ! Collect all the vertices in  $K_{N_1}$  which are at the other end of these edges coming out of  $x_2$ . Look at the complete graph (with edge colorings!) on those other vertices. It is a  $K_{N_2}$  whose internal edges again are colored red and/or blue. Also,  $N_2 \geq \frac{1}{2}(N_1 - 1)$ .



This picture tries to show the “evolution” from  $K_N$  to  $K_{N_1}$  to  $K_{N_2}$ . Symbolically we could write:

$$x_1 \text{ in } K_N \xrightarrow{C_1} K_{N_1} \text{ and } x_2 \text{ in } K_{N_1} \xrightarrow{C_2} K_{N_2}$$

where each time the number of vertices of the  $K_{\text{something}}$  is at least one-half the number of vertices of the previous graph. We’ll worry about the specific  $N$  needed later.

Let’s continue the process. We get

$$x_1 \xrightarrow{C_1} K_{N_1} \text{ and } x_2 \xrightarrow{C_2} K_{N_2} \text{ and } x_3 \xrightarrow{C_3} K_{N_3} \text{ and } x_4 \xrightarrow{C_4} K_{N_4} \text{ and } x_5 \xrightarrow{C_5} K_{N_5}$$

where we stop after five evolutions or reductions. Each of the colors (the  $C_j$ ’s) is either red or blue, and there are five of them. The pigeonhole principle again implies that at least one color (say **red** for specificity) must appear at least three times (since  $2 \cdot 2 = 4 < 5$  both can’t appear less than three times!). Again, let us assume for specificity that  $C_2$ ,  $C_3$ , and  $C_5$  are red. The logic will be similar for any other triple.

Now we will discover a red  $K_4$  in this  $K_N$ . Take  $y$  in  $K_{N_5}$ . Since  $C_5$  is red,  $(x_5, y)$  is red. But the creation of the new graphs is by shrinking, so  $y$  is also in  $K_{N_3}$  and so  $(x_3, y)$  is red. Similarly,  $y$  is in  $K_{N_2}$  and  $(x_2, y)$  is red. Since  $x_5$  is in  $K_{N_4}$ , it must be in  $K_{N_3}$  and therefore  $(x_3, x_5)$  is red.  $(x_2, x_3)$  and  $(x_2, x_5)$  are also red with the same arguments. So the vertices  $\{x_2, x_3, x_5, y\}$  and the associated edges give a monochromatic (in this case, red)  $K_5$ .

We must go back and check on the size of  $N$ . Since  $K_{N_5}$  is not empty (we need a  $y$  in it!), the size of  $K_{N_5}$  is at least 1. Then  $K_{N_4}$  should be at least 3, since we create  $K_{N_5}$  by

taking out  $x_5$  and being greedy.  $K_{N_3}$  must have at least 7 vertices, since we take out  $x_4$  and are greedy. Backwards again:  $K_{N_2}$  must have at least 15 vertices. And again:  $K_{N_1}$  must have at least 31 vertices. And finally:  $K_N$  must have at least 63 vertices.

If  $N \geq 63$ , then every 2-colored  $K_N$  contains a monochromatic  $K_4$ .

A logical scheme following this outline verifies a more general statement.

If  $N \geq 2^{2(n-1)} - 1$ , then every 2-colored  $K_N$  contains a monochromatic  $K_n$ .

## Lecture 17: Ramsey and five

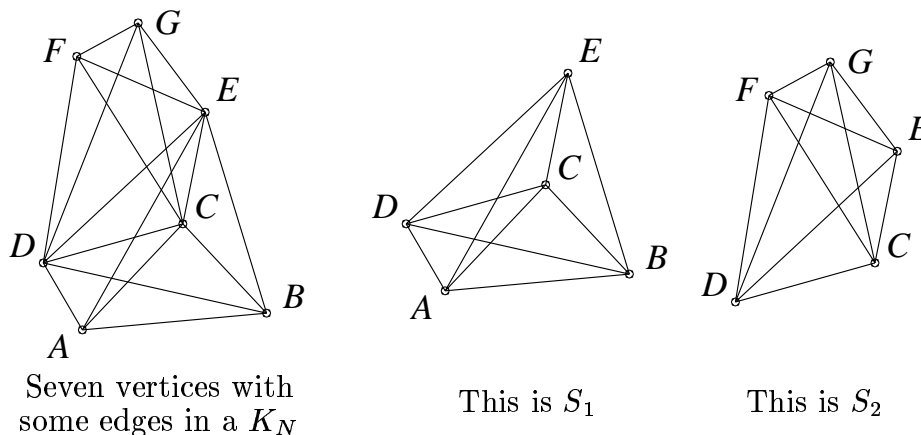
### 17.1 Definitions and $\mathcal{R}_5$

The  $n^{\text{th}}$  Ramsey number, which I'll write  $\mathcal{R}_n$ , is the "threshold number" for  $n$ . It is the smallest integer so that every 2-colored  $K_N$  with  $N \geq \mathcal{R}_n$  must have a monochromatic  $K_n$ . The procedure outlined above actually shows that  $\mathcal{R}_n$  exists, which is *not* obvious, and that  $\mathcal{R}_n \leq 2^{2^{(n-1)}} - 1$ .

We have shown that  $\mathcal{R}_4 \leq 63$  and our earlier example shows that  $\mathcal{R}_4 \geq 9$ . In fact,  $\mathcal{R}_4$ 's exact value is known: 18. This takes much more work.

The exact value of  $\mathcal{R}_5$  is *not* known\*. I will show you an *underestimate* of  $\mathcal{R}_5$  using a technique which is ludicrously simple (maybe) and fiendishly clever (really). It is relatively new and has become a powerful method for analyzing problems in many areas of theoretical computer science. It is also useful in aspects of theoretical physics, chemistry, and bioinformatics (computational biology).

Let's begin with an *uncolored*  $K_N$ . Let's flip coins to color the edges of this  $K_N$ . So we flip a fair coin (equal probability for heads and tails) repeatedly. Color an edge red if the coin lands heads, and blue if it shows a tail. Suppose that  $S$  is one of the  $K_5$ 's in  $K_N$ . What's the chance that  $S$  has all edges colored blue?  $S$  has 10 edges either because  $\binom{5}{2} = 10$  or because  $1 + 2 + 3 + 4 = 10$ . If the flips are fair and independent, the chance that  $S$  is all blue is  $\frac{1}{2^{10}}$ . Of course,  $S$  could also have all of its edges colored red, with the same probability. Therefore the chance that  $S$  is monochromatic is  $\frac{1}{2^9} = \frac{1}{512}$ . So far this is exactly like our analysis of trying to get three M&M's of one color. We computed the chance of the event that the three M&M's were green, then the chance they were red, etc., and finally we added these chances. This was simple because there was no overlapping of the random choices involved in these events. That is, there was no interaction between the events. But this situation is more complicated. An example may indicate the complexity.



What's shown is supposed to be seven vertices in a very big  $K_N$ . I haven't drawn every edge in order to keep the picture simpler. One  $K_5$  is  $S_1$  with vertex set  $\{A, B, C, D, E\}$ , and another is  $S_2$  with vertex set  $\{C, D, E, F, G\}$ . Each  $K_5$  has 10 edges.  $S_1$  and  $S_2$

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\* The best current information seems to be that  $\mathcal{R}_5$  is between 43 and 49.



share three edges:  $(C, D)$ ,  $(C, E)$ , and  $(D, E)$ . The chance that  $S_1$  is monochromatic is  $\frac{1}{2^9}$  and the chance that  $S_2$  is monochromatic is  $\frac{1}{2^9}$ . A careful computation shows that the chance that *both* of them are monochromatic is  $\frac{1}{2^{18}}$ . So the chance that at least one is monochromatic is less than the sum of the chances that each of them is monochromatic:

$$P(S_1 \text{ is monochromatic}) + P(S_2 \text{ is monochromatic}) = \\ P(\text{At least one is monochromatic}) + P(\text{Both are monochromatic}).$$

But we don't need to be so careful if we want a quick overestimate. We will forget the "fine structure" here and only use this idea, which overcounts:

The chance that at least one  $K_5$  is monochromatic will be overestimated by *the sum* of the chances that *any*  $K_5$  is monochromatic.

The number of  $K_5$ 's in  $K_N$  is  $\binom{N}{5} = \frac{N(N-1)(N-2)(N-3)(N-4)}{5!} = \frac{N^5 - 10N^4 + 35N^3 - 50N^2 + 24N}{120}$ . Each one separately has monochromatic probability  $\frac{1}{2^9} = \frac{1}{512}$ . So we can *overestimate* the probability of having at least one monochromatic  $K_5$  by\*  $\frac{N^5 - 10N^4 + 35N^3 - 50N^2 + 24N}{61440}$ . When  $N = 11$ , this number is approximately .90234, less than 1. If every 2-colored  $K_{11}$  had a monochromatic  $K_5$ , this number would have to be 1. So there must be some 2-colored  $K_{11}$  having no monochromatic  $K_5$ , and  $\mathcal{R}_5 \geq 12$ .

This underestimate of  $\mathcal{R}_5$ , showing the existence of a 2-colored  $K_{11}$  with no monochromatic  $K_5$ , is an example of "the probabilistic method". If this all seems clear, try the following exercise.

**Exercise** Find an explicit 2-coloring of  $K_{11}$  which has no monochromatic  $K_5$ .

**Comment** There are 462  $K_5$ 's in a  $K_{11}$ , and there are  $2^{55} = 36\,028\,797\,0189\,63968 \approx 3 \cdot 10^{21}$  2-colorings of  $K_{11}$ . This search space is very large. How can you find the 2-coloring requested? An exhaustive search may not quickly produce an example. Our computations assert that a "random" 2-coloring will be satisfactory about 10% of the time (actually more because we got an overestimate). Here's one strategy: first write a program to check for the presence of monochromatic  $K_5$ 's in a 2-colored  $K_{11}$ . Then create and test some random 2-colorings of  $K_{11}$  (you'll need at least a reasonable source of randomness!). You may fail to get a satisfactory coloring, but as you choose more and more random colorings, your chance of success *at least once* is close to certainty. That's because repeated failure for  $t$  attempts has approximate probability  $\leq (.9)^t$  which, as  $t$  grows, approaches 0 rapidly. For example, there's less than 1 chance in 40,000 of failure in 100 random attempts to create a suitable coloring. A direct search with carefully structured colorings, starting with all red edges, etc., may take a while to report success: your search may be stuck in an untypical corner of the exponentially large sample space, where there are many monochromatic  $K_5$ 's. "Random" will likely be faster than "deterministic" in this problem!

## 17.2 Thanks, a further remark, and a charming story

Conversations with David Galvin, János Komlós, and Jason Tedor helped me prepare this material. Dr. Komlós is a faculty member in mathematics and computer science at Rutgers and is one of the world's leading authorities on probabilistic methods. Mr. Tedor

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\*  $61440 = 120 \cdot 512$ .

is currently a graduate student in mathematics at Rutgers, while, as previously mentioned, Dr. Galvin recently graduated.

The methods given here for over- and underestimates of Ramsey numbers  $\mathcal{R}_n$  are, for  $n$  large, quite close to the best currently known! They can be used to show that  $\mathcal{R}_n$  is always between  $4^n$  and  $(\sqrt{2})^n$ .

Joel Spencer (in [6]) tells this anecdote about Paul Erdős, who was one of the greatest and strangest mathematicians of the twentieth century (see [8] and [9]). Erdős essentially invented the probabilistic method.

Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of  $\mathcal{R}_5$  or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for  $\mathcal{R}_6$ . In that case, he believes, we should attempt to destroy the aliens.

### 17.3 Bibliography

[1] Alexandre V. Borovik and Elena V. Bessonova, an undated webpage of the University of Manchester Institute of Science and Technology:  
<http://www.ma.umist.ac.uk/avb/Pigeon.html>

[2] Roger Day, a webpage from a summer 2001 course at Illinois State University:  
<http://www.math.ilstu.edu/~day/courses/old/305/pigeonholesamples.html>

Current information about Ramsey numbers is available here.

[3] Stanislaw Radziszowski, Dynamic Survey on *Small Ramsey Numbers*, 2001, *Electronic Journal of Mathematics*: <http://www.combinatorics.org/Surveys/>

There may be information about Ramsey theory and Ramsey numbers in introductory books about combinatorics and discrete mathematics, but the material is rather new. The following texts are the standard professional references on Ramsey theory.

[4] Noga Alon and Joel Spencer, *The Probabilistic Method*, second edition, 2000, John Wiley (320 pages, \$90, an advanced text).

[5] Ronald L. Graham, Joel H. Spencer, and Bruce Rothschild, *Ramsey Theory*, second edition, 1990, John Wiley (200 pages, \$145\*, an advanced text).

[6] Joel Spencer, *Ten Lectures on the Probabilistic Method*, 1987, Society for Industrial and Applied Mathematics (a paperback, 78 pages, \$29, perhaps a bit more accessible than the other two references).

There are several advanced computer science texts on randomized algorithms. One reviewer writes this about the text below: “For computer scientists, this is \*the\* reference work in randomized algorithms, by now a major paradigm of algorithms design.”

[7] Rajeev Motwani and Prabhakar Raghavan, *Randomized Algorithms*, 1995, Cambridge University Press (476 pages, \$55).

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\* This price is not a misprint.

Two books about Paul Erdős and his highly unusual life:

[8] Paul Hoffman, *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*, 1999, Little Brown (a paperback, 320 pages, \$10).

[9] Bruce Schechter, *My Brain Is Open: The Mathematical Journeys of Paul Erdős*, 2000, Touchstone Books (a paperback, 224 pages, \$10).