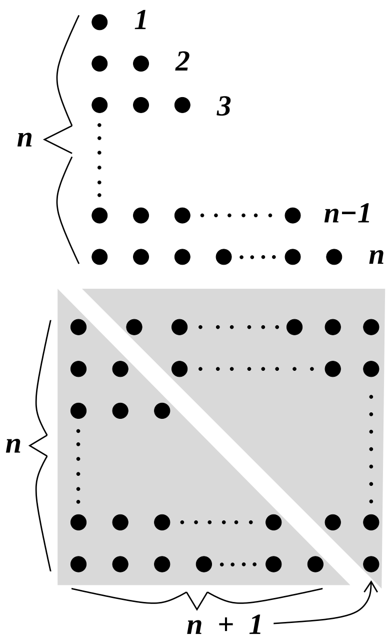
Discovering* a formula for the sum of integers with `Maple`

I'll describe what I hope to do at the beginning of the second meeting of our course.



First, let's get a formula for the sum of the first n positive integers. I'd like to compute $1+2+3+\ldots+n$. A sort of picture of that sum is shown to the right. You can see, I hope, why such sums are sometimes called *triangular numbers*. You may also have seen this as a special case of an arithmetic progression.

After you look closely at this picture, take another copy of it, turn it upside down, and slide it so that it "mates" with the original diagram. The two triangles just fit together. The way the second diagram is arranged, there are n rows and n+1 columns in a rectangular array. The total number of "dots" (the big dots!) is therefore $n(n+1)$. This represents *twice* the result we want so our sum is half of that.

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

These picture are cute and tricky and inspired.

What if I were not so clever, but I had hard-working computational help? That's the story for the next few minutes. Please forget the information I just told you about this problem!

Here's a function which is the sum of the integers from 1 to n:

v:=n->add(j,j=1..n);

and the output should be *v:=n→add(j,j=1..n)*

I'll check the function definition by typing, say, *v(3);* which should be the value of 1+2+3, so I hope 6 is the answer `Maple` returned.

We can get a bunch of values of *v* using the command

seq(v(q),q=1..10);

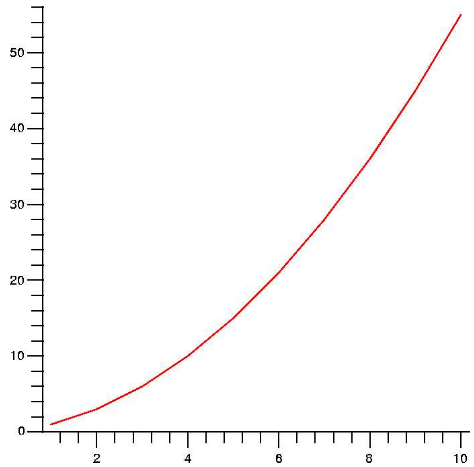
because what *seq* asks `Maple` to do is compute $v(q)$ for each integer as q runs from 1 to 10. The result should be 1, 3, 6, 10, 15, 21,28, 36, 45, 55 and these are the first 10 triangular numbers which I want to "understand". Maybe if I tried to plot them the picture would help me. Lots of brain power is devoted to vision, so pictures can be helpful. Let's plot some points. For example, since $v(6)=21$, the point $(6,21)$ would be on the graph I want. I need a collection of *ordered pairs*, and these can be created using the command

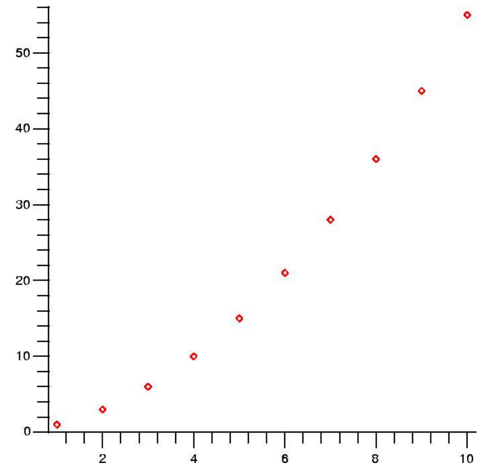
seq([q,v(q)],q=1..10);

The result should be [1,1], [2,3], [3,6], [4,10], [5,15], [6,21], [7,28], [8,36], [9,45], [10,55]

`Maple` uses [and] for "ordered pair" punctuation since the usual parentheses are needed in its own commands. I tried to *plot* these points. I first typed *plot(A);* which didn't work. I got `Error, (in plot) invalid arguments` and then I tried *plot([A]);* which gave me the curve (an interpolation of the points) shown on the left. The command *plot([A],style=point);* shows only the points and that picture is shown to the right.

* **and proving!**





Depending on my typing success (!) I could try some other experiments in order to "guess" at the rate of growth of the function v. For example, one traditional way biology or chemistry people sometimes analyzed data was to take logs and plot the result. In any case, I guess that v should be a quadratic function: $Aq^2 + Bq + C$.

We need to find good values for A and B and C. I want `Maple` to do the work. So define

w:=n->A∗n^2+B∗n+C;

which is a function involving 3 "unknowns" in its coefficients. `Maple` should have responded with $w{:=}n{\rightarrow}An^2{+}Bn{+}C$

I hoped that the 3 initial values of v would give me enough information to find A, B, and C. So I asked `Maple` to solve some equations: *solve({w(1)=v(1),w(2)=v(2),w(3)=v(3)});* which means I asked `Maple` to solve the equations $A + B + C = 1$ and $4A + 2B + C = 3$ and $9A + 3B + C = 6$ and I hope I got the solutions $A = \frac{1}{2}$, $B = \frac{1}{2}$, and $C = 0$. I substituted these into the formula for w and checked the results with a few cases. I never trust myself, and I certainly don't trust a machine, although the machine is probably more reliable computationally. How can we verify the formula for <u>all possible n's</u>? Let's think a bit. Here are the first few sums we're considering:

1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5, 1+2+3+4+5+6, etc.

I could describe the values of these sums in the following way:

1^{st} value=1, 2^{nd} value=1^{st} value+2, 3^{rd} value=2^{nd} value+3; 4^{th} value=3^{rd} value+4, etc.

where I can be rather more explicit about what "etc." means with some algebra: the $(n+1)^{\text{st}}$ value should be the sum of the n^{th} value and $n+1$. Our candidate formula is $w(n) = \frac{1}{2}n^2 + \frac{1}{2}n$. It is valid for $n = 1$ and $n = 2$ and $n = 3$. We certainly can check it for other n's. But if we verify that the sum of $w(n)$ and $n+1$ is the same as $w(n+1)$, that will show the correctness of the equation $w(n) = v(n)$ is *always* inherited from 1 and 2 and 3. I do no computations: `Maple` does them. So look at

w(n+1)-w(n);

and examine the result. We may need to push `Maple` a bit with *expand(%))* or *simplify(%))* after asking about $w(n+1) - w(n)$. If the result is eventually $n+1$ then we (`Maple` and a person!) have proved $1 + 2 + 3 + \ldots + n = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2}$ for all positive integers: an infinite number of statements verified by a finite machine.