Experimental Math: one of the top 50 Byrne Seminars in the fall 2008 semester!

Outline of how to use `Maple` to get a formula for the Fibonacci numbers

Here are commands we used to define and analyze the Fibonacci numbers and some discussion.

\> *fibon:=proc(n) if n=1 then 1 else if n=2 then 1 else fibon(n-1)+fibon(n-2) end if end if end proc;*

This defines the Fibonacci numbers. It uses **if** ... **then** ... **else** to get correct starting values.

\> *seq(fibon(n),n=1..10);*

I used this to look at the first 10 Fibonacci numbers and checked the procedure.

\> *growth:=solve(L^(n+2)=L^(n+1)+L^(n),L);*

This instruction gets the growth constants for the Fibonacci sequence. These were weird numbers with $\sqrt{5}$'s involved, and, to me, at least, they are totally unexpected. The instruction assigns name *growth* to these numbers.

\> *Lp:=growth[1];Lm:=growth[2];*

I assigned the two constants (I dropped the 0 value!) to Lp and Lm.

\> *v:=n->A*Lp^n+B*Lm^n;*

This is the model I used to try to get a formula for the Fibonacci numbers. How can we "guess" a good model? So much can be said and written about that, but one effective answer is this: if the model works, then the guess is good. The parameters A and B need to be determined.

\> *seq(fibon(n)=v(n),n=1..2);*

This displays the two equations in two unknowns needed to get the correct values of the parameters.

\> *coefficients:=solve({%});*

So we get the values of the parameters, and the name *coefficients* is given to the pair of equations defining these values (yes, this is a complicated phrase!).

\> *v1:=n->subs(coefficients,v(n));*

A new function, *v1* is defined. If you type *v1(n);* you'll see the resulting formula. If you type *seq(v1(n),n=1..10)* you should get the first 10 Fibonacci numbers. Although that's not *logically* necessary, I feel good when I can match these numbers with the sequence I started with! Actually, you may need to type *seq(expand(v1(n)),n=1..10)* to encourage `Maple` to do some arithmetic.

\> *v1(n+2)-v1(n+1)-v1(n);*

Now this checks (we hope!) that the formula satisfies the recurrence relation, and it is therefore correct for <u>all</u> n's. The `simplify(%)` command didn't help much, but `expand(%)` produced 0 as the answer, so the sequence produced by the formula is equal to the Fibonacci sequence <u>always</u>.

Problems for groups of students
with comments, clues, and perhaps some irrelevance

Solution team members Desai, Hard, Kalia, Nickelson

Question #1 Define a sequence recursively by these rules: the first element, a_1, is **1**, the second element, a_2, is **2**, and for $n > 2$, $a_n = \mathbf{3}a_{n-1} + \mathbf{4}a_{n-2}$. (What could be easier than something depending on 1, 2, 3, 4?) For your information, here are the first ten terms of this sequence.

$$1, 2, 10, 38, 154, 614, 2458, 9830, 39322, 157286$$

Here is what you should do:

1. Write a `Maple` procedure with name `tiger` which produces this sequence. Check that `seq(tiger(n),n=1..10)` actually produces the numbers above.
2. Find the growth constants and then *guess* a formula involving "unknown" coefficients for this sequence. This should be very similar to what we did for the Fibonacci numbers.
3. Find suitable values for the unknown coefficients using `solve` appropriately, and plug them into the formula that you have guessed. Then check that the first 10 values of the formula are the numbers above.
4. Check that your formula satisfies the recurrence (use `expand` or `simplify` – whichever works!). You have *proved* your formula always gives the values of the `tiger` sequence. You have tamed the `tiger`!

Comment Yes, you can cheat, but please don't. This sequence is in Sloane's *Encyclopedia*: it is A122117, and the first information about it was entered in 2006, and the entry was further modified as recently as July, 2008.

Problems for groups of students

with comments, clues, and perhaps some irrelevance

Solution team members Amin, Esmet, Flores, Patel

Question #2 Define a sequence recursively as follows: the first element, a_1, is **1**, the second element, a_2, is **2**, the third element, a_3 is **3**, and for $n > 3$, $a_n = \mathbf{3}a_{n-1} + \mathbf{6}a_{n-2} - \mathbf{8}a_{n-3}$. (What could be easier than 1, 2, 3 for the start of the sequence? Of course, the recursion seems (!) to be horrible, but when you work through this problem, you will see why it is actually very easy to handle.) For your information, here are the first ten terms of this sequence.

$$1, 2, 3, 13, 41, 177, 673, 2753, 10881, 43777$$

Here is what you should do:

1. Write a `Maple` procedure with name `lion` which produces this sequence. You will need three `if` ... `then` ... `else` ... statements in your procedure because there are three values given at the start of the sequence. Check that `seq(lion(n),n=1..10)` actually produces the numbers above.
2. Find the growth constants and then *guess* a formula involving "unknown" coefficients for this sequence. Here you can imitate what we did for the Fibonacci numbers, but things will be slightly more complicated – there will be three growth constants because this recurrence has three terms. Then *guess* a formula involving "unknown" coefficients for this sequence.
3. Find suitable values for the unknown coefficients using `solve` appropriately, and plug them into the formula that you have guessed. Then check that the first 10 values of the formula are the numbers above.
4. Check that your formula satisfies the recurrence (use `expand` or `simplify` – whichever works!). You have *proved* your formula always gives the values of the `lion` sequence. The `lion` is now quite predictable.

Comment I don't know any simple way to "cheat" and guess the details of the answer. In particular, this sequence does *not* seem to be in Sloane's *Encyclopedia*.

Byrne Seminar, slightly exciting stuff! More Experimental Math.

Problems for groups of students

with comments, clues, and perhaps some irrelevance

Solution team members Bae, Fridman, Vyas, Wu

Question #3 So here is a slight change from the Fibonacci sequence. Define a sequence recursively as follows: the first term, a_1, is 1, the second term, a_2, is 1, and for $n > 2$, $a_n = -\, a_{n-1} + a_{n-2}$. The only change from the Fibonacci rule is the minus sign. For your information, here are the first ten terms of this sequence.

$$1, 1, 0, 1, -1, 2, -3, 5, -8, 13$$

Here is what you should do:

1. Write a `Maple` procedure with name `mouse` which produces this sequence. Check that `seq(mouse(n),n=1..10)` actually produces the numbers above.
2. Find the growth constants and then *guess* a formula involving "unknown" coefficients for this sequence. This should be very similar to what we did for the Fibonacci numbers.
3. Find suitable values for the unknown coefficients using `solve` appropriately, and plug them into the formula that you have guessed. Then check that the first 10 values of the formula are the numbers above.
4. Check that your formula satisfies the recurrence (use `expand` or `simplify` – whichever works!). You have *proved* your formula always gives the values of the `mouse` sequence. You now have `mouse` thoroughly understood.

Comment Since you have `mouse` thoroughly understood, maybe you can tell me why the terms (after the first few) seem to be the Fibonacci numbers with alternating signs. Is there an obvious reason?*

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* For those of you who haven't taken or aren't taking a math course, you should be informed that the most irritating word to hear in such a course is *clearly.* This could mean either the reason is too darn complicated (possible), or it could mean that the instructor doesn't understand the situation well enough to explain what's going on (perhaps more likely), or, even more likely, things are not clear. So is there some way to explain the apparent coincidence here other than "clearly"?

Problems for groups of students

with comments, clues, and perhaps some irrelevance

Solution team members Brissac, Campbell, Shah, Zimmerman

Question #4 Here is a modification of the Fibonacci sequence, and some modification of the theoretical model will get a formula. Define a sequence with these rules: the first term, a_1, is 1, the second term, a_2, is 1, and for $n > 2$, $a_n = a_{n-1} + a_{n-2} + 1$. So we add 1 at each step. For your information, here are the first ten terms of this sequence.

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109$$

Here is what you should do:

1. Write a `Maple` procedure with name `toad` which produces this sequence. Check that `seq(toad(n),n=1..10)` actually produces the numbers above.
2. Find the growth constants and then *guess* a formula involving "unknown" coefficients for this sequence. Here you can imitate what we did for the Fibonacci numbers, but things will be more complicated because of the "+1" in the recurrence rule. You may need a different type of formula. Look below at the **Lessons learned** because in an experiment, you should be guided partly by what has already worked, and modify that to fit current circumstances.
3. Find suitable values for the unknown coefficients using `solve` appropriately, and plug them into the formula that you have guessed. Then check that the first 10 values of the formula are the numbers above.
4. Check that your formula satisfies the recurrence (use `expand` or `simplify` – whichever works!). You have *proved* your formula always gives the values of the `toad` sequence. You have *proved* your formula always gives the values of the `toad` sequence. You now have `toad` thoroughly understood.

Comment Yes, you can cheat, but please don't. This sequence is in Sloane's *Encyclopedia*: it is A001595, and the first information about it was entered in 2002, and the entry was further modified as recently as January, 2008. The entry states that members of the sequence are sometimes called *Leonardo numbers*.

Lessons learned During our third meeting, we looked at $a_n = 3a_{n-1} + 1$. The multiplicative constant is 3, and I guessed that the formula would be some constant multiplied by powers of 3 and maybe another constant added. So it would look like $A3^n + B$. This worked. $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ in the formula matched the sequence.
In our fourth meeting, we looked at $a_n = 5a_{n-1} - 3$. The 5 made me think if there is multiplication by 5 each time, the formula should probably involve 5^n. The -3 suggested modifying the formula with some additive constant. The model we tried was $A5^n + B$. The (absurd?) values $A = \frac{1}{20}$ and $B = \frac{3}{4}$ in the formula matched the sequence.
In the `toad` sequence, there are two growth constants and the $+1$. So you should create a model for the function which uses these *three* pieces of information. That's *three* pieces.

Problems for groups of students

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Solution team members Firminich, Kim, Noone, Weiner

Question #5 Define a sequence recursively by these rules: the first element, a_1, is 1, the second element, a_2, is 2, and for $n > 2$, $a_n = 2a_{n-1} + a_{n-2}$. So this is really just a slight change from the Fibonacci numbers (the (1,1) start is replaced by (1,2) and the recurrence is slightly different). For your information, here are the first ten terms of this sequence.

$$1, 2, 5, 12, 29, 70, 169, 408, 985, 2378$$

Here is what you should do:

1. Write a `Maple` procedure with name `horse` which produces this sequence. Check that `seq(horse(n),n=1..10)` actually produces the numbers above.
2. Find the growth constants and then *guess* a formula involving "unknown" coefficients for this sequence. This should be very similar to what we did for the Fibonacci numbers.
3. Find suitable values for the unknown coefficients using `solve` appropriately, and plug them into the formula that you have guessed. Then check that the first 10 values of the formula are the numbers above.
4. Check that your formula satisfies the recurrence (use `expand` or `simplify` – whichever works!). You have *proved* your formula always gives the values of the `horse` sequence. The `horse` has been ridden!

Comment I cheated and looked this up in Sloane's *Encyclopedia*. The sequence does appear there. It is sequence A000129. The numbers in the sequence, he writes, are "also called lambda numbers." The discussion is very long, and, interestingly, the formula which you get by solving this problem does *not* appear in the discussion. I don't understand why, since Sloane is a really clever person. Perhaps he should be told?
Also, look at 3 consecutive numbers in this sequence, say 5 and 12 and 29. Then $12^2 = 144$ and $5 \cdot 29 = 145$. These numbers differ by 1. Let's try another triple, say 408 and 985 and 2378. Then $985^2 = 970225$ while $408 \cdot 2378 = 970224$. You can check that this is not a mystical accident by using your formula together with appropriate `expand` and/or `simplify` commands, just as we did with the Fibonacci numbers.