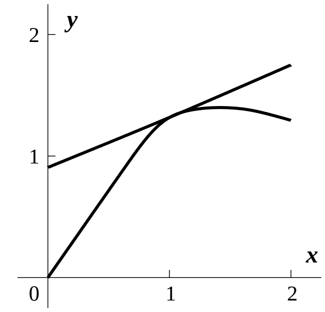
(10) 1. Suppose $f(x) = \dfrac{1}{x+2}$. Use the **definition of derivative** to find $f'(x)$. **Answer** Since $f(x) = \dfrac{1}{x+2}$, $f(x+h) = \dfrac{1}{x+h+2}$ and $f(x+h) - f(x) = \dfrac{1}{x+h+2} - \dfrac{1}{x+2} = \dfrac{(x+2)-(x+h+2)}{(x+h+2)(x+2)} = \dfrac{-h}{(x+h+2)(x+2)}$. Therefore $\dfrac{f(x+h)-f(x)}{h} = \dfrac{-1}{(x+h+2)(x+2)}$ and the limit of this as $h \to 0$ is $\dfrac{-1}{(x+2)^2}$, which is $f'(x)$.

(12) 2. a) Find the equation of a line tangent to $y = \dfrac{4x}{2+x^2}$ when $x = 1$.

Answer When $x = 1$, $y = \dfrac{4 \cdot 1}{2+1^2} = \dfrac{4}{3}$. And $y' = \dfrac{4(2+x^2) - 2x(4x)}{(2+x^2)^2}$. When $x = 1$, $y' = \dfrac{4(2+1^2) - 2(4)}{(2+1^2)^2} = \dfrac{4}{9}$. Therefore the tangent line has equation $y - \dfrac{4}{3} = \dfrac{4}{9}(x-1)$.

b) Sketch the graph of $y = \dfrac{4x}{2+x^2}$ and the tangent line found in a) on the axes below. **Answer** A picture is to the right.



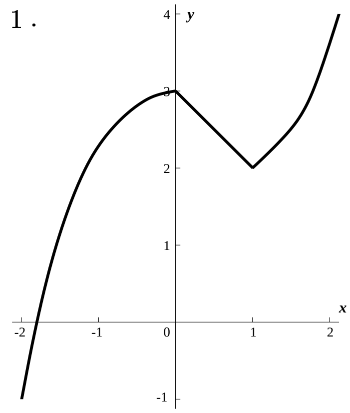
(14) 3. Suppose that the function $f(x)$ is described by $f(x) = \begin{cases} 3 - x^2 & \text{if } x < 0 \\ Ax + B & \text{if } 0 \le x \le 1 \\ 2^x & \text{if } 1 < x \end{cases}$.

a) Find A and B so that $f(x)$ is continuous for all numbers. Briefly explain your answer.

Answer $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3 - x^2 = 3$, and continuity implies this should equal $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} Ax + B = B$. So B must be 3. Also $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} Ax + B = A + B$, and again continuity makes this the same as $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2^x = 2$. Therefore $A + B = 2$ and since $B = 3$, A must be -1.

b) Sketch $y = f(x)$ on the axes given for the values of A and B found in a) when x is in the interval $[-2, 2]$.

Answer On the interval $[-2, 0]$, the graph of $y = 3 - x^2$ is drawn. On $[0, 1]$, the graph of $y = -x + 3$ is drawn. On $[1, 2]$, the graph of $y = 2^x$ is drawn. These curves all meet and the graph shows a continuous function.



(20) 4. Evaluate the indicated limits exactly. Give evidence to support your answers.

a) $\lim_{x \to 1} \dfrac{x^2 + 2x - 3}{x - 1}$

Answer Since $x^2 + 2x - 3 = (x+3)(x-1)$, $\dfrac{x^2+2x-3}{x-1}$ is $x + 3$ away from 1. Therefore we need $\lim_{x \to 1} x + 3$ which is 4.

b) $\lim_{x \to 2^+} \dfrac{|x-1| - 1}{|x-2|}$

Answer As $x \to 2^+$, we know $x > 2$. Therefore $x - 2 > 0$ and $|x-2| = x - 2$. Also certainly $|x-1| = x - 1$ when $x > 2$ so that $\dfrac{|x-1|-1}{|x-2|} = \dfrac{x-1-1}{x-2} = 1$ for $x > 2$. The value of the limit is 1.

c) $\lim_{x \to 3} \dfrac{\sqrt{x} - \sqrt{3}}{x - 3}$

Answer Multiply $\dfrac{\sqrt{x}-\sqrt{3}}{x-3}$ by $\dfrac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}$ and get $\dfrac{x-3}{(x-3)\cdot(\sqrt{x}+\sqrt{3})}$ which away from 3 is $\dfrac{1}{\sqrt{x}+\sqrt{3}}$. This has limit $\dfrac{1}{2\sqrt{3}}$ as $x \to 3$.

d) $\lim_{x \to 4} \dfrac{3x - 2}{\cos(\pi x)}$

Answer This function is continuous at 4, so its limit is its value at 4: $\dfrac{3 \cdot 4 - 2}{\cos(\pi 4)} = \dfrac{10}{\cos(4\pi)} = 10$.

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(8) 5. Suppose that $f(x) = x^5 + 3\cos\big(K(x^2)\big)$ where K is an unknown constant.

a) Briefly explain why $f(2)$ must be positive.

Answer We know that values of cosine must be between -1 and 1, and therefore $3\cos\big(K(2^2)\big)$ must be between -3 and 3. Since 2^5 is 32, the value of $f(2)$ is between 29 and 35. So $f(2)$ must be positive.

b) Briefly explain why $f(-2)$ must be negative.

Answer We know that values of cosine must be between -1 and 1, and therefore $3\cos\big(K(-2)^2\big)$ must be between -3 and 3. Since $(-2)^5$ is -32, the value of $f(2)$ is between -35 and -29. So $f(2)$ must be positive.

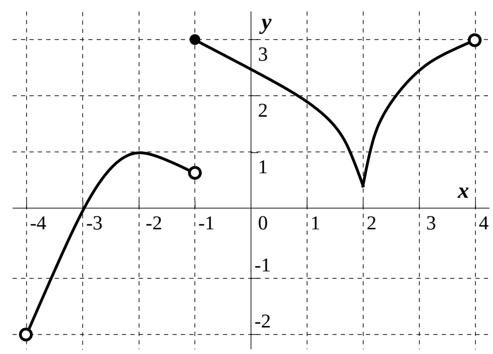
c) Briefly explain why the equation $f(x) = 0$ must have a solution, and specify an interval on the x-axis in which a solution can be found.

Answer $f(x)$ is a continuous function and $f(-2)$ is negative and $f(2)$ is positive. Therefore, by the Intermediate Value Theorem, $f(x) = 0$ must have at least one root inside the interval $[-2, 2]$.

(8) 6. What is the domain of $f(x) = \dfrac{\ln x + \sqrt{4-x}}{\sin x}$? Explain your answer algebraically.

Answer Certainly $x > 0$ so that $\ln x$ will be defined. Also $\sqrt{4-x}$ will be defined only when $4 - x \geq 0$ which is when $4 \geq x$. Therefore the top of the fraction defining $f(x)$ restricts the domain to $0 < x \leq 4$. The bottom, $\sin x$, should not be 0. The only x in $0 < x \leq 4$ for which $\sin x = 0$ is $x = \pi$. Therefore the domain is the two intervals $0 < x < \pi$ and $\pi < x \leq 4$.

(18) 7. In this problem the function $f(x)$ has domain all x's between -4 and 4: $-4 < x < 4$. A graph of $y = f(x)$ is displayed below. Answer the following questions as well as you can based on the information in the graph.



a) What is the range (the collection of values) of $f(x)$?

Answer Horizontal lines with $y =$ a constant intersect the graph for x's in the interval $-2 < x \leq 3$, which is the range.

b) For which x is $f(x)$ *not* continuous?

Answer Only at $x = -1$ where $f(x)$ has a jump.

c) For which x is $f(x) = 0$?

Answer $x = -3$

d) For which x is $f(x) > 0$?

Answer $-3 < x < 4$

e) For which x is $f(x)$ *not* differentiable?

Answer $x = -1$ (where the function is not continuous and therefore can't be differentiable) and $x = 2$, where the graph has a corner.

f) For which x is $f'(x) = 0$?

Answer $x = -2$

g) For which x is $f'(x) > 0$?

Answer The two intervals $-4 < x < -2$ and $2 < x < 4$

(10) 8. a) If $f(x) = \dfrac{1 - e^x}{x^2 + 1}$, what is $f'(x)$?

Answer $f'(x) = \dfrac{(-e^x)(x^2+1) - (2x)(1-e^x)}{(x^2+1)^2}$

b) If $f(x) = (2x + 3\cos x)(x^4 - x^2)$, what is $f'(x)$?

Answer $f'(x) = (2 - 3\sin x)(x^4 - x^2) + (2x + 3\cos x)(4x^3 - 2x)$

c) Suppose that $g(x)$ is a differentiable function and that $g(1) = 2$, $g'(1) = -3$, and $g''(1) = 4$. What is the value of the second derivative of $f(x) = x^3 g(x)$ when $x = 1$?

Answer Since $f(x) = x^3 g(x)$, I know that $f'(x) = 3x^2 g(x) + x^3 g'(x)$ and then $f''(x) = 6xg(x) + 3x^2 g'(x) + 3x^2 g'(x) + x^3 g''(x)$. The Product Rule has been used three times, once for computing $f'(x)$ and twice for computing $f''(x)$. Thus $f''(1) = 6 \cdot 1 g(1) + 3 \cdot 1^2 g'(1) + 3 \cdot 1^2 g'(1) + 1^3 g''(1) = 6 \cdot 2 + 3 \cdot (-3) + 3 \cdot (-3) + 4 = -2$.