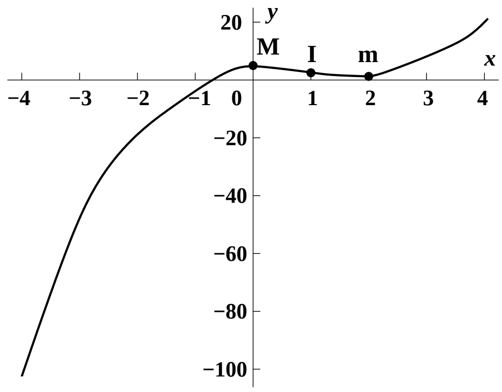
(15) 1. Suppose $f(x) = x^3 - 3x^2 + 5$. Be sure to answer all parts of this question.

a) Find all critical numbers of $f(x)$. Describe the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing.

Answer $f'(x) = 3x^2 - 6x$, so critical numbers are where $3x(x-2) = 0$ and that is $x = 0$ or $x = 2$. $f'(x) > 0$ for $(-\infty, 0)$ and $(2, \infty)$ so $f(x)$ is increasing there. $f'(x) < 0$ for $(0, 2)$ so $f(x)$ is decreasing there.

b) Find all inflection points of $f(x)$. Describe the intervals where $f(x)$ is concave up and where $f(x)$ is concave down.

Answer $f''(x) = 6x - 6$, and $f''(x)$ changes sign at $x = 1$ where $f''(x) = 0$. So $(1, f(1)) = (1, 3)$ is the only inflection point. $f(x)$ is concave down in $(-\infty, 1)$ and is concave up in $(1, \infty)$.

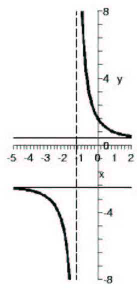


c) Sketch a graph of $y = f(x)$ on the axes given. Label any relative maxima with an **M** on your graph. Label any relative minima with an **m** on your graph. Label any inflection points with an **I** on your graph.

Note The units on the horizontal and vertical axes are different.

(12) 2. Find the equations of all vertical and horizontal asymptotes of $f(x) = \frac{3e^x+5}{7e^x-2}$. Numerical approximations are *not* acceptable.

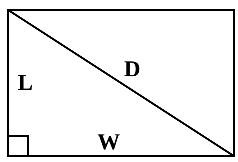
Answer Since the limits as $x \to +\infty$ of both the top and bottom are $+\infty$, we use l'Hopital's rule: $\lim\limits_{x\to+\infty} \frac{3e^x+5}{7e^x-2} \overset{\text{l'H}}{=} \lim\limits_{x\to+\infty} \frac{3e^x}{7e^x} = \frac{3}{7}$. Therefore $y = \frac{3}{7}$ is a horizontal asymptote. As $x \to -\infty$, $e^x \to 0$, so $\lim\limits_{x\to-\infty} \frac{3e^x+5}{7e^x-2} = -\frac{5}{2}$. Therefore $y = -\frac{5}{2}$ is a horizontal asymptote. The bottom, $7e^x - 2$, is 0 exactly when $e^x = \frac{2}{7}$ and this is $x = \ln\left(\frac{2}{7}\right)$. This is a vertical asymptote, since as $x \to \ln\left(\frac{2}{7}\right)^{\pm}$, $f(x) \to \pm\infty$.



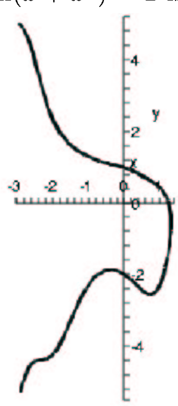
Note To the right is a graph of $y = f(x)$ and its asymptotes which may help you "see" the asymptotes.

(10) 3. At a certain time, the length of a rectangle is 3 feet and its width is 5 feet. At that time, the length is *increasing* at .4 feet per second, and the width is *decreasing* at .5 feet per second. What is the length of the diagonal of the rectangle at that time? At that time, how fast is the length of the rectangle's diagonal changing? Is this length increasing or decreasing?

Answer Suppose D is the diagonal length. Then $D^2 = L^2 + W^2$ so that $D^2 = 3^2 + 5^2 = 34$ and D must be $\sqrt{34}$. We differentiate and get $2D\frac{dD}{dt} = 2L\frac{dL}{dt} + 2W\frac{dW}{dt}$. At the "certain time," this is $2\sqrt{34}\frac{dD}{dt} = 2(3)(+.4) + 2(5)(-.5) = -2.6$ so that $\frac{dD}{dt} = -\frac{1.3}{\sqrt{34}}$. The length is decreasing.



(12) 4. A graph of a portion of the curve described implicitly by the equation $y^2 + x^3 + y + y\sin(x + x^2) = 2$ is shown to the right.



a) Find the coordinates of the two points of intersection of this curve with the y-axis (where $x = 0$).

Answer When $x = 0$ the equation becomes $y^2 + y = 2$ or $y^2 + y - 2 = 0$ or $(y+2)(y-1) = 0$. The coordinates are $(0, 1)$ and $(0, -2)$, which seem to agree with the picture.

b) Are the tangent lines to $y^2 + x^3 + y + y\sin(x + x^2) = 2$ at the two points where the curve intersects the y-axis parallel? Use calculus to answer this question.

Answer We need y', the derivative. Let's $\frac{d}{dx}$ the equation. Using the product and chain rules, we get: $2yy' + 3x^2 + y' + y'\sin(x + x^2) + y\cos(x + x^2)(1 + 2x) = 0$. On the y-axis, $x = 0$, so this equation becomes (with $\sin(0) = 0$ and $\cos(0) = 1$) just $2yy' + y' + y = 0$ or $y' = -\frac{y}{2y+1}$. At $(0, 1)$, $y' = -\frac{1}{3}$, and at $(0, -2)$, $y' = -\frac{(-2)}{2(-2)+1} = -\frac{2}{3}$, The derivatives, which are the slopes of the tangent lines, are not equal so the tangent lines are *not* parallel.

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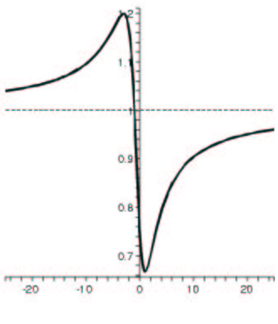
(8) 5. Suppose $f(x) = \sqrt{2 + 7x^3}$.
a) Compute $f(1)$. **Answer** $f(1) = \sqrt{2 + 7 \cdot 1^3} = 3$.
b) Compute $f'(1)$. **Answer** Since $f(x) = (2+7x^3)^{1/2}$, $f'(x) = \frac{1}{2}(2+7x^3)^{-1/2}21x^2$ and $f'(1) = \frac{1}{2} \cdot \frac{1}{3} \cdot 21 = \frac{7}{2}$.
c) Find an approximate value for $f(1.08)$ using the differential or tangent line approximation for $f(x)$ and your answers to a) and b). **Answer** $f(x + h) \approx f(x) + f'(x)h$. Here $x = 1$ and $h = .08$, so $f(1.08) \approx f(1) + f'(1)(.08) = 3 + \frac{7}{2}(.08) = 3.28$. ($f(1.08) \approx 3.289070385$ so the linearization answer is good.)

(8) 6. Suppose $g(x)$ is a differentiable function, and we know that $g(1) = 2$, $g'(1) = 3$, and $g''(1) = -1$. Define the function $h(x)$ by the equation $h(x) = g(2x^3 - x^2)$. Compute $h(1)$, $h'(1)$, and $h''(1)$.
Answer Since $h(x) = g(2x^3 - x^2)$, $h(1) = g(2 \cdot 1^3 - 1^2) = g(1) = 2$. The chain rule gives $h'(x) = g'(2x^3 - x^2)(6x^2 - 2x)$ so $h'(1) = g'(2 \cdot 1^3 - 1^2)(6 \cdot 1^2 - 2 \cdot 1) = g'(1)(4) = 3 \cdot 4 = 12$. Both the chain rule and the product rule are needed to compute $h''(x)$. The result is $h''(x) = g''(2x^3 - x^2)(6x^2 - 2x)^2 + g'(2x^3 - x^2)(12x - 2)$. When $x = 1$, $h''(1) = g''(1)(4)^2 + g'(1)(10) = (-1) \cdot 4^2 + 3(10) = 14$.

(18) 7. Suppose $f(x) = \frac{x^2+3}{x^2+x+4}$.
a) What is the domain of $f(x)$? Why?
Answer The domain is all real numbers. The roots of $x^2 + x + 4 = 0$ involve $\sqrt{1^2 - 4(1)(4)}$, the square root of a negative number. So there are no real roots, and the formula defining $f(x)$ is valid for all real x's.
b) What is $\lim_{x\to+\infty} f(x)$? Why? What is $\lim_{x\to-\infty} f(x)$? Why?
Answer As $x \to \pm\infty$, both the top and bottom of the formula defining $f(x)$ get large ($+\infty$) so the desired limits are eligible for l'Hopital's rule. Therefore $\lim_{x\to\infty} \frac{x^2+3}{x^2+x+4} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{2x}{2x+1}$ but this is also $\frac{\infty}{\infty}$. Again using l'Hopital's rule, we get $\lim_{x\to\infty} \frac{2}{2} = 1$. The limit as $x \to -\infty$ has the same result. So both limits are 1.
c) Use calculus to find any relative maximum and relative minimum values of $f(x)$.
Answer $f'(x) = \frac{(2x)(x^2+x+4)-(x^2+3)(2x+1)}{(x^2+x+4)^2}$. To see where $f'(x) = 0$ we need to analyze the top: $(2x)(x^2+x+4) - (x^2+3)(2x+1) = 2x^3 + 2x^2 + 8x - (2x^3 + x^2 + 6x + 3) = x^2 + 2x - 3$. This is 0 when $(x+3)(x-1) = 0$ or when $x = -3$ and when $x = 1$. Now $f(-3) = \frac{(-3)^2+3}{(-3)^2-3+4} = \frac{12}{10}$ and $f(1) = \frac{1^2+3}{1^2+1+4} = \frac{4}{6}$. Probably there's a relative max at -3 and a relative min at 1. Since $f'(x) = \frac{(x+3)(x-1)}{\text{STUFF}^2}$ the sign of $f'(x)$ depends on $(x+3)(x-1)$. For $x < -3$, $f'(x) > 0$ so $f(x)$ is increasing. A sign change occurs at -3 and another at 1. So $f(x)$ is decreasing between -3 and 1 and then increasing after 1. Therefore we have justified the preceding designations of relative max and min.
d) The *range* of a function is the collection of all possible values of the function. What is the exact range of $f(x)$? Explain your answer using calculus.
Answer The range is all numbers between $f(1) = \frac{4}{6}$ and $f(-3) = \frac{12}{10}$: the closed interval $\left[\frac{4}{6}, \frac{12}{10}\right]$. $f(x)$ decreases from $\frac{12}{10}$ to $\frac{4}{6}$ between -3 and 1. Since $f(x)$ is continuous, the Intermediate Value Theorem implies that the range is at least $\left[\frac{4}{6}, \frac{12}{10}\right]$. From 1 to $+\infty$, by previous parts of the problem, $f(x)$ decreases from $\frac{12}{10}$ to 1 (notice that 1 is between $\frac{4}{6}$ and $\frac{12}{10}$). From $-\infty$ to -3, $f(x)$ increases from 1 to $\frac{4}{6}$.
Note To the right is part of the graph of $y = f(x)$ and its horizontal asymptote, $y = 1$. Look at the values on the axes, please. The graph may help the explanation.



(9) 8. Suppose the sum of two non-negative numbers is 15. Verify using calculus that the largest the product of the square of one multiplied by the other can be is 500.
Answer The constraints are $x + y = 15$ with both x and $y \geq 0$. We want to maximize x^2y. Certainly $y = 15 - x$, so we need to maximize $f(x) = x^2(15 - x)$ for $0 \leq x \leq 15$. Notice that $f(0) = 0$ and $f(15) = 0$ so a positive maximum will be attained at a critical number inside the interval. But $f'(x) = 2x(15 - x) + x^2 = x(30 - 3x)$. The only interior critical number is $x = 10$, and there we have $f(10) = 10^2(15-5) = 100(5) = 500$. We're done since this is positive and larger than the values at the endpoints.

(8) 9. Suppose that $F(x) = (2x^3 + 8)^{12} - (7 - 3x^5)^{15}$.
a) Compute $F'(x)$. **Answer** $F'(x) = 12(2x^3 + 8)^{11}(6x^2) - 15(7 - 3x^5)^{14}(-15x^4)$.
b) Use your answer to a) and calculus to explain briefly why $F(200)$ is less than $F(400)$.
Answer For $x > 0$, $F'(x) = 12(2x^3 + 8)^{11}(6x^2) + 15(7 - 3x^5)^{14}(15x^4)$ is positive, and therefore $F(x)$ is increasing for $x > 0$. Thus $F(200) < F(400)$. ($F(200) \approx 10^{180}$ and $F(400) \approx 10^{202}$ so we are correct.)