(20) 1. Evaluate the indicated limits exactly. Give evidence to support your answers.

a) $\lim_{x\to -2} \frac{x^2-x-6}{x+2}$

Answer $x^2+x-6=(x-3)(x+2)$. The quotient is $x-3$ which has limit -5 as $x\to -2$.

b) $\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x}$

Answer Multiply top and bottom by $\sqrt{3+x}+\sqrt{3}$. The top then becomes $(3+x)-3=x$ which cancels the x on the bottom to leave $\frac{1}{\sqrt{3+x}+\sqrt{3}}$. This has limit $\frac{1}{2\sqrt{3}}$ as $x\to 0$.

c) $\lim_{x\to 1^-} \frac{3x+\ln x}{x}$

Answer This function is continuous at 1 and its value there is $\frac{(3\cdot 1+\ln 1)}{1}=3$ since $\ln 1$ is 0.

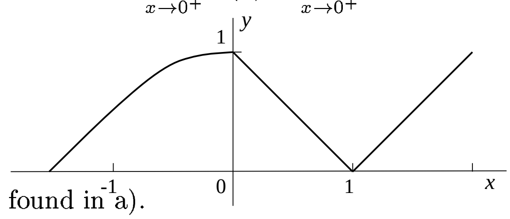
d) $\lim_{x\to 0^+} \frac{2}{\sin x}$

Answer The bottom is a small positive number when $x\to 0^+$ while the top is fixed at $+2$. The quotient therefore is a large positive number, and the answer is $+\infty$.

(14) 2. Suppose that the function F is described by $F(x)=\begin{cases}\cos x, & \text{if } -\frac{\pi}{2}\le x\le 0\\ Ax+B & \text{if } 0<x\le 1\\ x-1 & \text{if } 1<x\le 2\end{cases}$.

a) Find A and B so that F is continuous for all numbers in its domain. Briefly explain your answer.

Answer The only discontinuities can occur at 0 and 1. The left- and right-hand limits of F at 0 must agree with F's value at 0. $F(0)=\cos 0=1$. $\lim_{x\to 0^-}F(x)=\lim_{x\to 0^-}\cos x=\cos 0=1$. $\lim_{x\to 0^+}F(x)=\lim_{x\to 0^+}Ax+B=A\cdot 0+B=B$. So $B=1$. The left- and right-hand limits of F at 1 must agree with F's value at 1. $F(1)=A\cdot 1+B=A+B$. $\lim_{x\to 1^-}F(x)=\lim_{x\to 1^-}Ax+B=A\cdot 1+B=A+B$. $\lim_{x\to 1^+}F(x)=\lim_{x\to 1^+}x-1=1-1=0$. So $A+B=0$. Since $B=1$, A is -1.

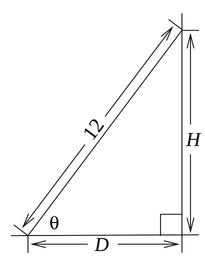


b) Graph $y=F(x)$ on the axes given for the values of A and B found in a).

(12) 3. A ladder which is 12 feet long has one end on flat ground and the other end on the vertical wall of a building. H is the height from the ground to the point at which the ladder touches the building. D is the distance between the bottom of the ladder and the bottom of the wall. θ is the acute angle between the ladder and the ground.

a) Write H as a function of D: that is, a formula involving D and no other <u>variable</u>. What is the domain of this function?

Answer Pythagoras gives $(12)^2=H^2+D^2$ so $H=\sqrt{(12)^2-D^2}$. The domain is $0\le D\le 12$.



b) Write H as a function of θ: that is, a formula involving θ and no other <u>variable</u>. What is the domain of this function?

Answer $\sin\theta=\frac{H}{12}$ so $H=12\sin\theta$. The domain is $0\le\theta\le\frac{\pi}{2}$. It would have been a *better* problem if I had asked for θ as a function of H, I think.

(12) 4. a) Write the definition of derivative as a limit and *use this definition* to find the derivative of $f(x)=\frac{2}{x+3}$.

Answer $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$. When $\Delta x\neq 0$ we see: $\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\frac{2}{x+\Delta x+3}-\frac{2}{x+3}}{\Delta x}=\frac{\frac{2\cdot(x+3)-2\cdot(x+\Delta x+3)}{(x+\Delta x+3)(x+3)}}{\Delta x}=\frac{\frac{-2\Delta x}{(x+\Delta x+3)(x+3)}}{\Delta x}=\frac{-2}{(x+\Delta x+3)(x+3)}$. This last algebraic form allows us to find the limit: $f'(x)=\lim_{\Delta x\to 0}\frac{-2}{(x+\Delta x+3)(x+3)}=\frac{-2}{(x+3)^2}$.

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b) Use your answer to a) to find an equation for the line tangent to the curve $y = \dfrac{2}{x+3}$ when $x = 2$.
Answer The slope of the tangent line when $x = 2$ is $f'(1) = \frac{-2}{(2+3)^2} = -\frac{2}{25}$ and one point on the tangent line is $(1, f(1))$ which is $(1, \frac{2}{2+3}) = (1, \frac{2}{5})$. An equation for the line is therefore $y - \frac{2}{5} = -\frac{2}{25}(x-2)$, which is a fine answer. It can be converted to slope-intercept form: $y = -\frac{2}{25}x + \frac{14}{25}$ but it doesn't have to be.

(9) 5. Do *not* algebraically simplify the answers in this problem!
a) If $f(x) = -8x^7 + 4x^4 + 24$, what is $f'(x)$?
Answer $f'(x) = -8(7x^6) + 4(4x^3) + 0$.

b) If $g(x) = \dfrac{x^2+1}{2e^x - 3}$, what is $g'(x)$?
Answer $g'(x) = \dfrac{(2x)(2e^x - 3) - (2e^x)(x^2+1)}{(2e^x-3)^2}$.

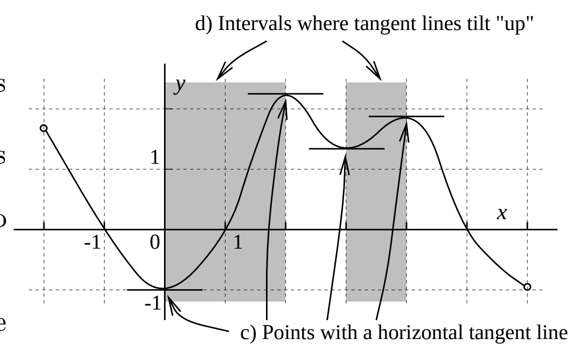
c) If $h(x) = \sqrt{3 - \sin x}$, what is $h'(x)$?
Answer $h'(x) = \frac{1}{2}(3 - \sin x)^{-(1/2)}(-\cos x)$.

(8) 6. k is a differentiable function and the following is known about k and its derivatives.

$$k(1) = 2 \quad k'(1) = -3 \quad k''(1) = 4$$

Suppose that $j(x) = k(e^x)$ (this is a composition). Compute $j(0)$ and $j'(0)$ and $j''(0)$. Give the exact answer in each case.
Answer $j(0) = k(e^0) = k(1) = 2$. $j'(x) = k'(e^x)e^x$ by the chain rule, so $j'(0) = k'(e^0)e^0 = k'(1)\cdot 1 = -3\cdot 1 = -3$. Now use the chain rule *and* the product rule to get $k''(x) = k''(e^x)(e^x)^2 + k'(e^x)(e^x)$, so $j''(0) = k''(e^0)(e^0)^2 + k'(e^0)(e^0) = k''(1)(1)^2 + k'(1)\cdot 1 = 4\cdot 1^2 + (-3)\cdot 1 = 1$.

(13) 7. Below is the graph of a differentiable function, Q, whose domain is $-2 < x < 6$. Use this graph to answer the following questions as accurately as you can.



The graph of $y = Q(x)$

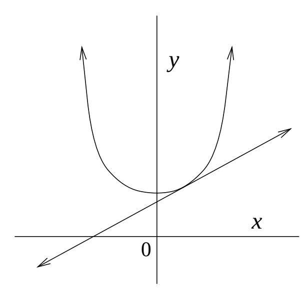
a) For which x's is $Q(x) = 0$?
Answer $Q(x) = 0$ when the graph of Q crosses the x-axis. This seems to occur three times: at $x = -1$ or 1 or 5.

b) What are the largest intervals on which $Q(x) > 0$ for all x's in each interval?
Answer The graph must be "above" the x-axis. This seems to occur in the two intervals $-2 < x < -1$ and $1 < x < 5$.

c) For which x's is $Q'(x) = 0$?
Answer The slope of the tangent line for these x's is 0, so the tangent line must be horizontal. This seems to occur four times: at $x = 0$ or 2 or 3 or 4.

d) What are the largest intervals on which $Q'(x) > 0$ for all x's in each interval?
Answer The tangent line must tilt "upwards" (the graph seems to have increasing y's as x increases): this seems to occur in two intervals: $0 < x < 2$ and $3 < x < 4$.

(12) 8. The line $y = \frac{1}{2}x + 3$ is tangent to the parabola $y = x^2 + B$. Use algebra and calculus to find the exact value of B.



Hint: the two formulas are equal at the tangent point, and the line is tangent to the curve at that point.
Answer If x_0 is the first coordinate of the tangent point, then $\frac{1}{2}x_0 + 3 = (x_0)^2 + B$. But the line must also be the tangent line, and the slope of the line tangent to $y = x^2 + B$ at x_0 is the derivative of $x^2 + B$ at x_0, which is $2x_0$. The slope of $y = \frac{1}{2}x + 3$ is always $\frac{1}{2}$ so $\frac{1}{2} = 2x_0$. Therefore x_0 is $\frac{1}{4}$ and $\frac{1}{2}x_0 + 3 = (x_0)^2 + B$ becomes $\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + 3 = \left(\frac{1}{4}\right)^2 + B$. so $B = 3 + \frac{1}{8} - \frac{1}{16}$ (a fine answer!). If you want, B is $3\frac{1}{16}$ or B is $\frac{49}{16}$ or B is 3.0625(!).