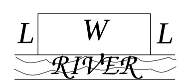
(12) 1. Locate the exact maximum and minimum values of the function $f(x) = 3x^5 - 5x^3$ on the closed interval $[-.5, 1.5]$ using calculus. Briefly explain using calculus why the values you give are the extreme values of the function on the closed interval. **Answer** $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1)$. $f$'s critical numbers are 0 and $\pm 1$. $-1$ isn't in our domain. Extrema of a continuous function in a closed interval must occur either at endpoints or at critical points, so extrema must occur at $-\frac{1}{2}$ or $\frac{3}{2}$ or 0 or 1. Here are the values of $f$ at these points: $f(-.5) = .53125$, $f(0) = 0$, $f(1) = -2$, and $f(1.5) = 5.90625$. The exact maximum value of $f$ is 5.90625 and the exact minimum value is $-2$.

(12) 2. Suppose a point is moving along the graph of $y = x\sin x$ so that $\dfrac{dx}{dt}$ is 2 centimeters per second.

a) Find $\dfrac{dy}{dt}$ when $x = \pi$. **Answer** $\dfrac{dy}{dt} = \dfrac{dx}{dt}\sin x + x\cos x\dfrac{dx}{dt}$ using the product rule and the chain rule. When $x = \pi$, $\dfrac{dy}{dt} = 2\cdot\sin\pi + \pi(\cos\pi)\cdot 2 = 2\cdot 0 + \pi\cdot -1\cdot 2 = -2\pi$.

b) If $Q = x^2 + y^2$, is $Q$ increasing or decreasing when $x = \pi$? **Answer** Differentiate $x^2 + y^2$ with respect to time and get $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2\pi(2) + 2y(-2\pi)$. But $y = \pi\sin\pi = 0$ when $x = \pi$ so that $Q' = 4\pi$ when $x = \pi$. Since $Q'$ is positive, $Q$ is increasing when $x = \pi$.

(16) 3. A dairy farmer plans to fence in a rectangular pasture adjacent to a straight river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?




Explain briefly why you have found the *least* amount of fencing.

**Answer** Suppose the sides of the pasture perpendicular to the river have length $L$, and the side parallel to the river has length $W$. Then $LW = 180{,}000$ (the constraint) and we wish to minimize the total fence length, $T = 2L + W$ (the objective function). We can use the constraint to write $T$ as a function of one variable: $T = 2L + \frac{180{,}000}{L}$. Note that the domain of $T$ is $0 < L < \infty$. $T' = 2 - \frac{180{,}000}{L^2}$ and this is 0 when $L^2 = 90{,}000$ so $L = 300$. The dimensions needed are 300 meters perpendicular to the river and $180{,}000/300 = 600$ meters parallel to the river.

Why is this a minimum? Here are some possible reasons (just one is requested!):

**The function level** $\lim_{L\to 0^+} T = +\infty$ and $\lim_{L\to+\infty} T = +\infty$ using the formula for $T$. Therefore $L = 300$ must be a minimum.

THE ZEROTH(?) DERIVATIVE TEST

**The first derivative level** Since $T' = 2 - \frac{180{,}000}{L^2}$, when $L$ is to the left of 300 (between 0 and 300) $T' < 0$ and when $L$ is very large positive, $T' > 0$. Therefore $T$ decreases to the left of 300 and increases to the right of 300. So $T$ must have a minimum at $L = 300$.

THE FIRST DERIVATIVE TEST

**The second derivative level** $T'' = \frac{360{,}000}{L^3}$ which is positive when $L = 300$. Therefore $T$ is concave up at the critical number $L = 300$, so $T$ must have a minimum at $L = 300$.

THE SECOND DERIVATIVE TEST

(20) 4. Suppose that $F(x) = \dfrac{x^2-3}{1-x^2}$.

a) What is the domain of $F$? Where is $F(x) = 0$? Find any vertical asymptotes of the graph of $F$. Find any horizontal asymptotes of the graph of $F$.

**Answer** The domain of $F$ is all numbers except $\pm 1$ (where $x^2 - 1$ is 0). $F$ is 0 when $x = \pm\sqrt{3}$. Writing $F(x) = \frac{x^2-3}{(1-x)(1+x)}$ may help you find vertical asymptotes. Then the limits near $\pm 1$ should be clearer. For example, $\lim_{x\to -1^-} F(x)$ think of $x$ as close to $-1$ but *less than* $-1$. Then $x^2 - 3$ will be close to $-2$, $1 - x$ will be close to 2, and $1 + x$ will be a small negative number. Putting all this together yields $\lim_{x\to -1^-} F(x) = +\infty$. So $x = -1$ is a vertical asymptote. Similarly, $x = +1$ is a vertical asymptote. When $x \to \pm\infty$, the dominant term in the top of $F$ is $x^2$ and in the bottom of $F$ is $-x^2$. Therefore $\lim_{x\to\pm\infty} F(x) = -1$ and $y = -1$ is a horizontal asymptote. The symmetry of $F$ ($F(x) = F(-x)$) certainly helps to check results in this problem.

b) Compute $F'(x)$ carefully. Where is $F'(x) = 0$? Analyze the behavior of $F$ near each critical number you have found: declare whether it is a relative minimum, relative maximum, or neither, and briefly explain your reasoning.

**Answer** $F'(x) = \frac{2x\cdot(1-x^2) - (-2x)\cdot(x^2-3)}{(1-x^2)^2} = \frac{-4x}{(1-x^2)^2}$. $F'$ is 0 only when $x = 0$. $F'$ is positive for $x < 0$ and negative for $x > 0$ so $F$ has a relative maximum at 0.

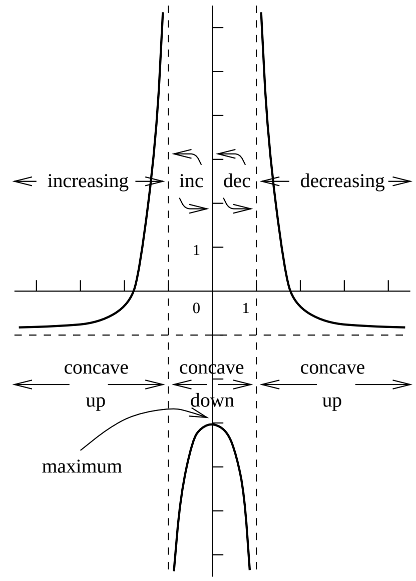
c) Compute $F''(x)$ carefully and simplify the result. The following statement is true: **$F$ has no inflection points but $F$'s concavity is different in different intervals.** Briefly explain why each assertion in this statement is correct using your algebraic expression for $F''(x)$ and facts connecting the second derivative and concavity.

**Answer** $F''(x) = \frac{-4(1-x^2)^2-(-4x)\cdot 2(1-x^2)\cdot(-2x)}{((1-x^2)^2)^2} = \frac{-4-12x^2}{(1-x^2)^3}$. This computation may be irritating! If you got through it safely, notice that the top is *always* negative and *never* zero. Therefore this differentiable function cannot have any inflection points. However, the bottom *can* change sign depending on $x$. If $-1 < x < 1$ then the bottom is positive so $F''$ is negative. There the function is concave down. If $x < -1$ or $x > +1$ the sign of the bottom is negative, and then $F'' > 0$. In those intervals the function is concave up.

d) Sketch a graph of $y = F(x)$. Label on your graph each of the following:

- Intervals where $F$ is increasing and where it is decreasing.
- Intervals where $F$ is concave up and where it is concave down.
- Any relative maxima or minima of $F$.

Also include the graphs of any horizontal or vertical asymptotes.

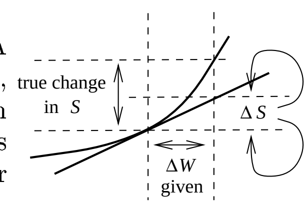



(12) 5. Assume that the surface area in square meters $S$ of a baby gnu is related to its weight in kilograms $W$ by the formula $S = 5W^{3/2} + 3W^2$.

a) What is the surface area when the weight is 16 kilograms? **Answer** $S = 1088$.

b) Use the differential or tangent line approximation to describe what percentage increase in surface area will result from a 5% increase in weight. Briefly explain with a graph or with further computation using calculus why your answer is greater than or less than the exact answer, which is approximately 9.4685%.

**Answer** We know that $\Delta S \approx S'\Delta W$. $S' = 5\cdot\frac{3}{2}W^{1/2} + 6W$ so when $W = 16$, $S' = 126$. A 5% increase in W is $.05 \cdot 16 = .8$ kilograms. If $\Delta W = .8$, then $\Delta S \approx 126 \cdot .8 = 100.8$ and, as a percentage increase, this is $(100.8/1088)\cdot 100 \approx 9.2647\%$. This answer is less than the exact answer because $S$ is concave up, and the tangent line lies below the graph. $S$ is concave up because $S'' = 5\cdot\frac{3}{2}\cdot\frac{1}{2}W^{-1/2} + 6$ is positive. Alternatively, a graph of $S$ near $W = 16$ looks like the picture shown.



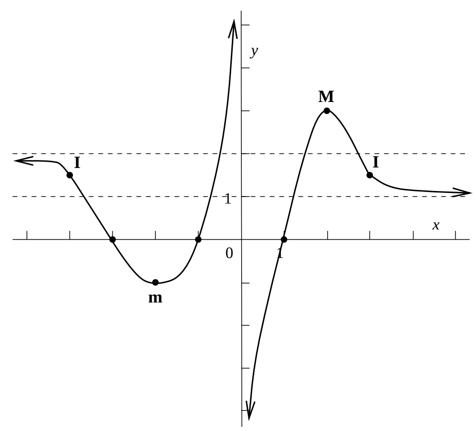

(14) 6. The domain of a differentiable function $H$ is all real numbers except 0. $H$ has the following properties.

- $\lim_{x\to 0^-} H(x) = +\infty$ and $\lim_{x\to 0^+} H(x) = -\infty$.
- $\lim_{x\to -\infty} H(x) = 2$ and $\lim_{x\to +\infty} H(x) = 1$.
- $H(x) = 0$ only when $x = -3$ and $x = -1$ and $x = 1$.
- $H'(x) = 0$ only when $x = -2$ and $x = 2$, and $H(-2) = -1$ and $H(2) = 3$.
- $H''(x) = 0$ only when $x = -4$ and $x = 3$, and $H(-4) = 1.5$ and $H(3) = 1.5$.

Sketch a graph of $H$. Label any relative maxima on the graph with **M** and any relative minima with **m** and any inflection points with **I**. Write the equations of any horizontal asymptotes and any vertical asymptotes.

Equations of any horizontal asymptotes: **Answer** $y = 1$ and $y = 2$.

Equations of any vertical asymptotes: **Answer** $x = 0$.




(14) 7. The equation $x + Dy^2 = e^{xy} + C$ defines a curve for certain values of the constants $C$ and $D$.

a) Find $\dfrac{dy}{dx}$ in terms of $x$ and $y$ and $C$ and $D$ (remember in this computation that $C$ and $D$ are unspecified constants!).

**Answer** $\frac{d}{dx}$ the equation above, remembering to use the chain rule and the product rule. The result is: $1 + 2Dyy' = e^{xy}(1\cdot y + xy') + 0$ (the derivative of a constant is 0). We solve for $y'$: $2Dyy' - e^{xy}xy' = e^{xy}y - 1$ so $y' = \frac{e^{xy}y-1}{2Dy-e^{xy}x}$.

b) Find values of $C$ and $D$ so that the line $y = 3x + 2$ is tangent to the curve at the point $(0, 2)$.

**Answer** "Plug in" $x = 0$ and $y = 2$ into the original equation and get $4D = 1 + C$ because $e^0 = 1$. Now "plug in" $x = 0$ and $y = 2$ into the equation for $y'$ but realize that $y'$ must be 3 because $y = \boxed{3}x + 2$ is tangent to the curve at $(0, 2)$. Therefore $3 = \frac{2-1}{4D} = \frac{1}{4D}$ so $D = \frac{1}{12}$. The first equation then becomes $4\cdot\frac{1}{12} = 1 + C$, so $C = -\frac{2}{3}$.