(8) 1. A point is moving along the graph of the function $y = \sin x$ so that $\frac{dx}{dt}$ is 2 centimeters per second. Find y and $\frac{dy}{dt}$ when $x = \frac{\pi}{6}$.

Answer $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Use the chain rule: $\frac{dy}{dt} = (\cos x)\frac{dx}{dt}$. If $x = \frac{\pi}{6}$, $\frac{dy}{dt} = \left(\cos\left(\frac{\pi}{6}\right)\right)\cdot 2 = \left(\frac{\sqrt{3}}{2}\right)2 = \sqrt{3}$.

(10) 2. Find the indicated limits. Give evidence to support your answers.

a) $\lim_{x\to+\infty}\frac{3x+2}{2x-1}$ **Answer** $\frac{3x+2}{2x-1} = \frac{3x+2}{2x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \frac{3+\frac{2}{x}}{2-\frac{1}{x}}$. But $\frac{2}{x}$ and $\frac{1}{x}$ both $\to 0$ as $x \to +\infty$, so the limit is $\frac{3}{2}$.

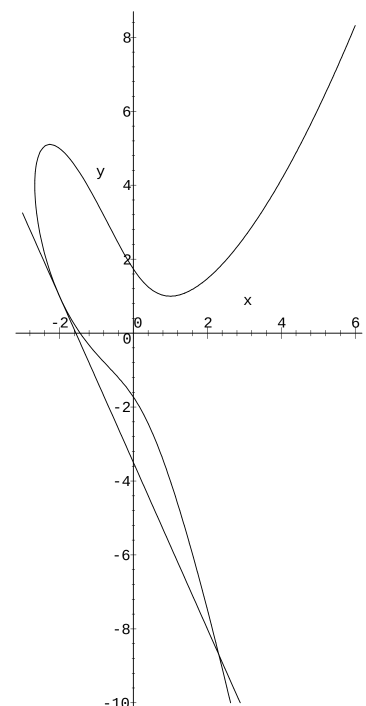
b) $\lim_{x\to\frac{1}{2}^+}\frac{3x+2}{2x-1}$ **Answer** As $x \to \frac{1}{2}^+$, $2x-1 \to 2\left(\frac{1}{2}\right)-1 = 0$ and $2x-1 > 0$. Also $3x+2 \to 3\left(\frac{1}{2}\right)+2 = \frac{7}{2}$. The quotient is therefore $\frac{\approx\frac{7}{2}}{\text{small positive \#}}$ which is a large positive number, so the limit is $+\infty$.

(12) 3. Find all relative maximum and minimum values of the function $f(x) = (x^2-3)e^x$. Briefly explain your answers using calculus.

Answer Relative extrema must occur at critical numbers. Here $f'(x) = 2xe^x + (x^2-3)e^x = (x^2+2x-3)e^x$. $f'(x)$ exists everywhere, so critical numbers occur where $f'(x) = 0$. But $e^x > 0$ for all x, so we need to know where $x^2+2x-3 = (x+3)(x-1)$ is 0. That happens at $x = -3$ or $x = 1$. $f'(x)$ changes sign at -3 and 1. Its sign is positive for $x < -3$ and $x > 1$ and negative for $-3 < x < 1$. Therefore f is increasing to the left of -3 and decreasing to the right of -3, and $f(-3) = ((-3)^2-3)e^{-3} \approx .3$ is a relative maximum. Since f is decreasing to the right of 1 and increasing to the left of 1, $f(1) = ((1)^2-3)e^1 \approx -5.4$ is a relative minimum. f'' also gives information about the critical numbers. $f''(x) = (2x+2)e^x + (x^2+2x-3)e^x = (x^2+4x-1)e^x$ so that $f''(-3) = -4e^{-3} < 0$ and $f''(1) = 4e > 0$. Again we see f has a relative maximum at -3 and a relative minimum at 1.

Comment In this problem and in problems 5 and 6, please realize again that polynomials in the real world often don't factor neatly. An exam is *not* reality!

(14) 4. The program `Maple` displays this image when asked to graph the equation $y^2 = x^3 - 3xy + 3$. [In fact, what is shown here is the original picture together with the answer to part d).]



a) Verify by substitution that the point $P = (-2,1)$ is on the graph of the equation.

Answer At $(-2,1)$, $x^3 - 3xy + 3 = (-2)^3 - 3(-2)1 + 1 = -8 + 6 + 1 = 1$. But $y^2 = 1^2 = 1$ also, so $(-2,1)$ satisfies the equation.

b) Find $\frac{dy}{dx}$ in terms of y and x.

Answer Implicit differentiation with both the chain rule and the product rule give the following: $2yy' = 3x^2 - 3y - 3xy'$. We solve for y' in two steps. First, $2yy' + 3xy' = 3x^2 - 3y$ and then (since $2yy' + 3xy' = (2y+3x)y'$) $y' = \frac{3x^2-3y}{2y+3x}$

c) Find an equation for the line tangent to the graph at the point $P = (-2,1)$.

Answer At P, $y' = \frac{3x^2-3y}{2y+3x} = \frac{3(-2)^2-3(1)}{2(1)+3(-2)} = \frac{12-3}{2-6} = -\frac{9}{4}$. An equation of the tangent line must be $y - 1 = -\frac{9}{4}(x-(-2))$. This can be "simplified", if you must, to $y = -\frac{9}{4}x - \frac{7}{2}$.

d) Sketch this tangent line in the appropriate place on the image displayed.

Answer I had `Maple` draw this. The slope of the line drawn is negative. The approximate x-intercept of the tangent line also agrees with the previous answer.

(20) 5. Suppose $W(x) = -\frac{1}{2}x^2 + 6x - 5\ln x$.

a) Compute $W'(x)$ and $W''(x)$. Where are these functions equal to 0?

Answer $W'(x) = -x + 6 - \frac{5}{x}$. $W'(x) = 0$ when $-x + 6 - \frac{5}{x} = 0$ or when $-x^2 + 6x - 5 = 0$ or when $x^2 - 6x + 5 = 0$. But $x^2 - 6x + 5 = (x-5)(x-1)$ so $W'(x) = 0$ at 1 and 5. $W''(x) = -1 + \frac{5}{x^2}$. Now $-1 + \frac{5}{x^2} = 0$ when $x^2 = 5$. This occurs when $x = \pm\sqrt{5}$. Actually the single root of this equation which is in the domain of $W''(x)$ is $+\sqrt{5}$ because the domain of $W(x)$ (which has a logarithm in it!) is only positive x! Here either the answer $+\sqrt{5}$ or the answer $\pm\sqrt{5}$ will be accepted.

b) What is $\lim_{x\to 0^+} W(x)$?

Answer As $x \to 0$, certainly $-\frac{1}{2}x^2 + 6x \to 0$. Note that to consider $\ln x$ we must have $x > 0$. If $x \to 0^+$, then $\ln x \to -\infty$. We can put all the pieces together to conclude that $W(x) \to +\infty$ as $x \to 0^+$.

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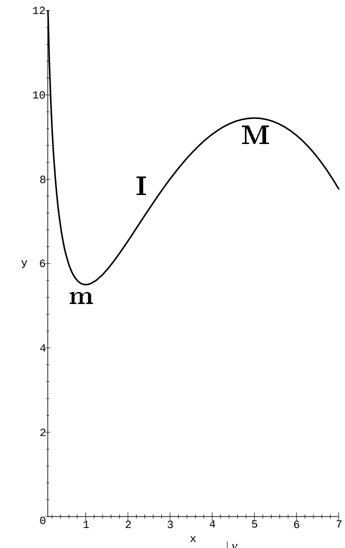
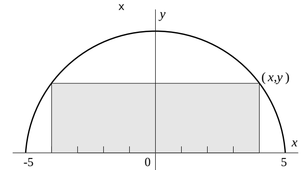
c) Sketch a graph on the axes given of $y = W(x)$ for x between 0 and 7.
- Label any relative maxima with an **M** on your graph.
- Label any relative minima with an **m** on your graph.
- Label any points of inflection with an **I** on your graph.

Answer Again I asked `Maple` to draw this picture for me. Then I labeled the "bumps", **M** and **m**, and the "wiggle", **I**.



- In what interval(s) on this graph is $W(x)$
  - increasing? **Answer** $1 \le x \le 5$
  - decreasing? **Answer** $0 < x \le 1$ & $5 \le x < 7$
  - concave up? **Answer** $0 < x \le \sqrt{5}$
  - concave down? **Answer** $\sqrt{5} \le x < 7$

(16) 6. A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum? Briefly explain using calculus why your answer gives a maximum.



Answer The rectangle's horizontal dimension (the length) is $2x$. Its vertical dimension (the width) is $\sqrt{25 - x^2}$. The rectangle's area is $A(x) = 2x\sqrt{25 - x^2}$. $A(x)$'s domain is $0 \le x \le 5$. Extreme values are either at critical numbers or at endpoints. $A(0) = 0$ and $A(5) = 0$. $A'(x) = 2\sqrt{25 - x^2} + 2x \cdot \frac{1}{2}(25 - x^2)^{-1/2}(-2x)$, which can be rewritten algebraically: $2\sqrt{25 - x^2} - \frac{2x^2}{\sqrt{25-x^2}} = \frac{2(25-x^2)-2x^2}{\sqrt{25-x^2}} = \frac{50-4x^2}{\sqrt{25-x^2}}$. $A'(x)$ doesn't exist at 5 in our domain (when the bottom is 0) but that's an endpoint and already considered. $A'(x) = 0$ when $50 - 4x^2 = 0$ which happens when $x = \frac{5}{\sqrt{2}}$ (choose the positive square root to be in our domain). $A(\frac{5}{\sqrt{2}}) = 2(\frac{5}{\sqrt{2}})\sqrt{25 - (\frac{5}{\sqrt{2}})^2}$ and this is a maximum (see the justification below, please!). The length, $2x$, is $5\sqrt{2}$, and the width, $\sqrt{25 - x^2}$, is $\sqrt{25 - (\frac{5}{\sqrt{2}})^2}$.

The function level The function $A(x)$ attains its maximum at either endpoints or critical numbers in this closed interval. Its value at the endpoints is 0, and its value at the only critical number is clearly positive. Therefore $A(x)$ must attain its maximum at the critical number.

THE ZERO$^{\text{th}}$ DERIVATIVE TEST

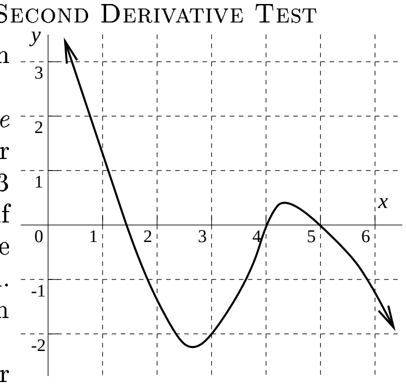
The first derivative level Since $A'(x) = \frac{50-4x^2}{\sqrt{25-x^2}}$ it is easy to see that for $0 \le x < \frac{5}{\sqrt{2}}$, $A'(x) > 0$ and for $\frac{5}{\sqrt{2}} < x \le 5$, $A'(x) < 0$. Therefore A is increasing to the left of $\frac{5}{\sqrt{2}}$ and decreasing to the right of $\frac{5}{\sqrt{2}}$. So $A(x)$ must have a maximum at $x = \frac{5}{\sqrt{2}}$.

THE FIRST DERIVATIVE TEST

The second derivative level We *can* compute $A''(x)$. It is $\frac{-8x\sqrt{25-x^2}-(50-4x^2)\frac{1}{2}(25-x^2)^{-1/2}(-2x)}{(\sqrt{25-x^2})^2}$. We only need the sign of this when $x = \frac{5}{\sqrt{2}}$. The $50 - 4x^2$ term vanishes at this value of x (!) so A'' at the critical number is $\frac{-8x\sqrt{25-x^2}}{(\sqrt{25-x^2})^2}$. There's one minus sign and everything else is positive. Since $A''(\frac{5}{\sqrt{2}}) < 0$, $A(x)$ is concave down and $A(x)$ must have a maximum.

THE SECOND DERIVATIVE TEST

(20) 7. To the right is a graph of $h'(x)$, the *derivative* of a function, h. Use this graph to answer the questions below.



$h'(x)$, the *derivative* of $h(x)$

a) Use information from the graph of $h'(x)$ to find the x where the *maximum value* of h in the interval $1 \le x \le 3$ will occur. Briefly explain using calculus why your answer is correct. **Answer** There's exactly one x (≈ 1.4) in the interval $1 \le x \le 3$ with $A'(x) = 0$. If $1 \le x < 1.4$, $A'(x) > 0$ so there $A(x)$ is increasing, and if $1.4 < x \le 3$, $A'(x) < 0$ so there $A(x)$ is decreasing. Therefore A has a relative maximum at 1.4, *and* $A(1.4) > A(x)$ if x is any *other* number in the interval. **Comment** We can't conclude that $A(x)$'s maximum value occurs at 1.4 only from the fact that $A(x)$ has a relative maximum at 1.4. More discussion is needed.

b) Suppose that $h(3) = 5$. Use information from the graph and the differential or tangent line approximation to find an approximate value of $h(3.04)$. Briefly explain using calculus and information from the graph why your approximation for $h(3.04)$ is greater than or less than the exact value of $h(3.04)$. **Answer** $h(x + \triangle x) \approx h(x) + h'(x)\triangle x$. Here $x = 3$ and $\triangle x = .04$. We're told that $h(3) = 5$ and the graph shows that $h'(3) = -2$, so $h(2.04) \approx 5 + (-2) \cdot (.04) = 4.92$. The difference between the function and its tangent line for small $\triangle x$'s is proportional to the second derivative, $h''(x)$. $h''(3)$ is the derivative of $h'(x)$ at $x = 3$. Tangent lines to the graph of $h'(x)$ near $x = 3$ have positive slope, so $h''(3) > 0$. The true value of $h(3.04)$ must be *greater than* 4.92 because $y = h(x)$ must be concave up near $x = 3$. Here's a picture of $h(x)$ and its tangent line near 3, illustrating that $h'(3) < 0$ and $h''(3) > 0$.