**640:151:04–06** **Answers to version A of the First Exam** **10/14/2006**

Here are answers that would earn full credit. Other methods may also be valid.

(12) 1. Compute the derivatives of the functions shown. In this problem, you may write only the answers and get full credit. *Please* do not "simplify" the answers!

a) $3+5x^3-12x^7$ **Answer** $0+5(3x^2)-12(7x^6)$.

b) $\frac{e^x+7}{x^2+1}$ **Answer** $\frac{(e^x+0)(x^2+1)-(e^x+7)(2x)}{(x^2+1)^2}$.

c) $\left(\frac{7}{x^2}+3\cos x\right)\left(5x^2+\sin x\right)$ **Answer** $\left(-7(2x^{-3})-3\sin x\right)\left(5x^2+\sin x\right)+\left(\frac{7}{x^2}+3\cos x\right)\left(5(2x^1)+\cos x\right)$.

(5) 2. Compute $F(2)$ and $F'(2)$ if $F(x)=x^3f(x)$ and $f$ is differentiable with $f(2)=-3$ and $f'(2)=-1$. Show supporting work for your answers. **Answer** $F(2)=2^3f(2)=8(-3)=-24$. $F'(x)=3x^2f(x)+x^3f'(x)$ so $F'(2)=3(2^2)(-3)+2^3(-1)=-44$.

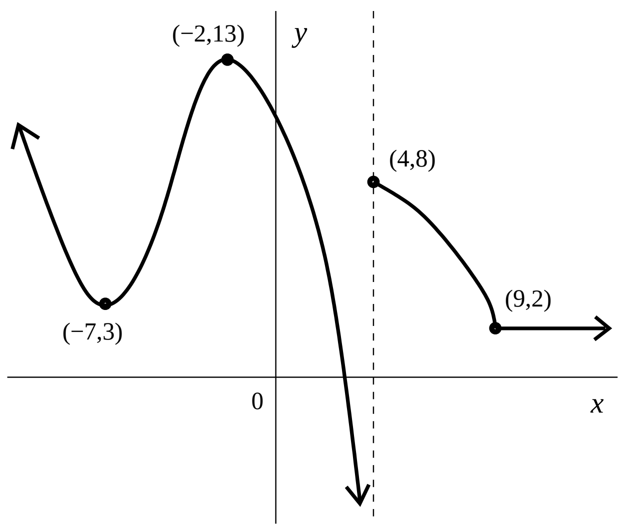
(15) 3. a) State the formal definition of the derivative, $f'(x)$, of the function $f(x)$. **Answer** $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

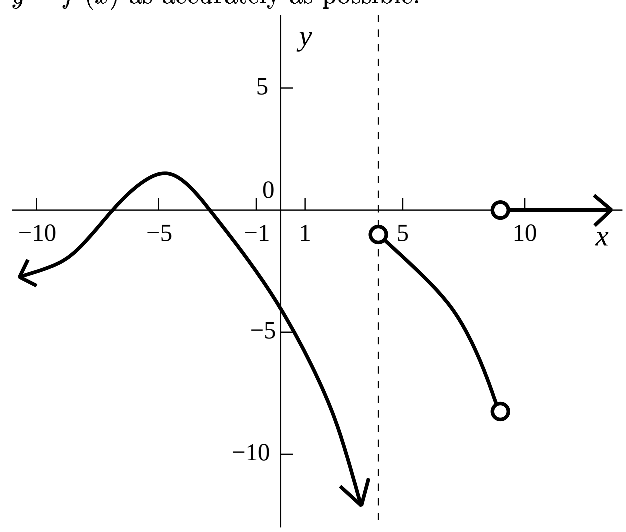
b) Use your answer to a) combined with algebraic manipulation and standard properties of limits to compute the derivative of $f(x)=\frac{x}{x^2+1}$. **Answer** $f(x+h)=\frac{x+h}{(x+h)^2+1}=\frac{x+h}{x^2+2xh+h^2+1}$ so that $f(x+h)-f(x)=$ $\frac{x+h}{x^2+2xh+h^2+1}-\frac{x}{x^2+1}=\frac{(x+h)(x^2+1)-(x^2+2xh+h^2+1)x}{(x^2+2xh+h^2+1)(x^2+1)}=\frac{(x^3+x^2h+x+h)-(x^3+2x^2h+xh^2+x)}{(x^2+2xh+h^2+1)(x^2+1)}=\frac{-x^2h-xh^2+h}{(x^2+2xh+h^2+1)(x^2+1)}$.

Therefore $\frac{f(x+h)-f(x)}{h}=\frac{\frac{-x^2h-xh^2+h}{(x^2+2xh+h^2+1)(x^2+1)}}{h}=\frac{-x^2-xh+1}{(x^2+2xh+h^2+1)(x^2+1)}$ As $h\to 0$, this $\to\frac{-x^2+1}{(x^2+1)(x^2+1)}=\frac{-x^2+1}{(x^2+1)^2}$ so $f'(x)=\frac{-x^2+1}{(x^2+1)^2}$. (You can check the answer with the Quotient Rule.)

(14) 4. Here is a graph of $y=f(x)$. [BELOW]

a) Use these axes to sketch a graph of $y=f'(x)$ as accurately as possible. [BELOW]




b) For which $x$'s can you conclude that $f$ is *not* continuous? **Answer** $x=4$.

c) For which $x$'s can you conclude that $f$ is *not* differentiable? **Answer** $x=4$ and $x=9$.

(16) 5. Compute these limits. Supporting work for each answer *must* be given to earn full credit.

**Comment** For the "experts": do **NOT** use l'Hospital's rule.

a) $\lim\limits_{x\to 3}\frac{x-3}{x^2-5x+6}$ **Answer** Since $x^2-5x+6=(x-3)(x-2)$, this limit is the same as $\lim\limits_{x\to 3}\frac{1}{x-2}$ which is 1 since $3-2\neq 0$.

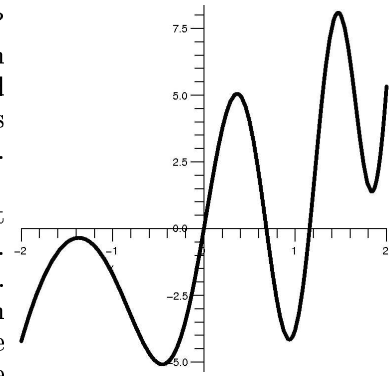
b) $\lim\limits_{x\to+\infty}\frac{x^3-9x+2}{-2x^3+4x^2}$ **Answer** Since $\frac{x^3-9x+2}{-2x^3+4x^2}=\frac{1-\frac{9}{x^2}+\frac{2}{x^3}}{-2+\frac{4}{x}}$ (divide top and bottom by $x^3$) and, as $x\to+\infty$, $\frac{1}{x}\to 0$, $\frac{1}{x^2}\to 0$, and $\frac{1}{x^3}\to 0$, the result is $\frac{1}{-2}$ or $-\frac{1}{2}$.

c) $\lim\limits_{x\to 0}\frac{4x^2-3x+2}{5e^{6x}}$ **Answer** Plug in $x=0$. The result is $\frac{4(0^2)-3(0)+2}{5e^0}=\frac{2}{5}$. Since the bottom is not 0, the function involved is continuous at 0 and the limit can be evaluated by "plugging in".

d) $\lim\limits_{x\to 10^-}\frac{x}{100-x^2}$. **Answer** As $x\to 10^-$, we know $x<10$ and $x$ is close to 10. Then $100-x^2$ will be small and positive. The top will be close to 10. Therefore as $x\to 10^-$, $\frac{x}{100-x^2}\approx\frac{10}{\text{small positive}}$, so the limit is $+\infty$.

(8) 6. a) Find numbers $K$ and $L$ so that $K \le 2+5\sin\left(x^2+4x\right) \le L$ for all numbers $x$. Give evidence to support your answer. **Answer** The values of sine are all in the interval $[-1,1]$ so the values of $2+5\sin(\text{anything})$ must be between $2+5(-1)=-3=K$ and $2+5(1)=7=L$.

b) Suppose $f(x)=x^3+2+5\sin\left(x^2+4x\right)$. What is the sign of $f(2)$? What is the sign of $f(-2)$? You may use your answers to a) to explain your answers here. **Answer** $(-2)^3=-8$ so $f(-2)$ is the sum of $-8$ and a number between $-3$ and 7. Therefore $-11 \le f(-2) \le -1$: $f(-2)$ is negative. $2^3=8$ so $f(2)$ is the sum of 8 and a number between $-3$ and 7. Therefore $5 \le f(2) \le 15$: $f(2)$ is positive.




c) Explain why the equation $f(x)=x^3+5\sin\left(x^2+4x\right)=0$ must have at least one solution. Give an interval in which this solution must be found. You must quote a specific result from this course, explaining its relevance. Your answers to b) may be useful here. **Answer** $f(x)$ is continuous on the interval $[-2,2]$ and the Intermediate Value Theorem applies. Since $f(-2)<0$ and $f(2)>0$, the Theorem states that there must be some number $w$ in the interval $[-2,2]$ so that $f(w)=0$. **Note** The graph above to the right shows $y=f(x)$ on $[-2,2]$ and gives additional evidence for solutions. The vertical and horizontal scales on the graph differ.

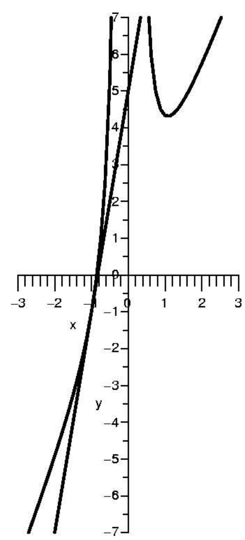
(16) 7. Suppose $f(x)=Ax+\frac{B}{x^2}$ where $A$ and $B$ are constants. Find values of $A$ and $B$ so that $y=6x+5$ is tangent to $y=f(x)$ when $x=-1$.

**Hint** Get information about $f(-1)$ and $f'(-1)$ from the given tangent line and then use this to find $A$ and $B$.

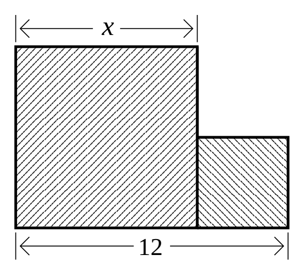
**Answer** Since $y=6x+5$ is tangent to $y=f(x)$ when $x=-1$, the line must pass through the point $(-1,f(-1))$. When $x=-1$, $6x+5=-1$ so $f(-1)=-1$. The slope of the line, 6, must be the slope of the line tangent to $y=f(x)$ at $x=-1$. But this is the derivative at $-1$ and therefore $f'(-1)=6$. Since $f(x)=Ax+\frac{B}{x^2}$, $f(-1)=A(-1)+\frac{B}{(-1)^2}=-A+B$. And $f'(x)=A-\frac{2B}{x^3}$, so that $f'(-1)=A+2B$.

$A$ and $B$ are the solutions of this system of two linear equations: $\begin{cases} -A+B=-1 \\ A+2B=6 \end{cases}$. If we add the equations, we get $3B=5$ so $B=\frac{5}{3}$. Either equation then gives $A=\frac{8}{3}$.

**Note** The graph to the right shows a part of $y=f(x)$ when $A=\frac{8}{3}$ and $B=\frac{5}{3}$ together with the line $y=6x+5$.




(14) 8. Two squares are placed so that their sides are touching, as shown. The sum of the lengths of one side of each square is 12 feet, as shown. Suppose the length of one side of one square is $x$ feet. a) Write a formula for $f(x)$, the sum of the total area of the two squares. What is the domain of this function when used to describe this problem? (The domain should be related to the problem statement.)




**Answer** The area of the left square is $x^2$. The side length of the right square must be $12-x$, so its area is $(12-x)^2$. The total area, which is $f(x)$, is the sum of the two areas. So $f(x)=x^2+(12-x)^2$. The domain is $0 \le x \le 12$ since squares can't have *negative* side lengths, and any side length can't be greater than 12. (I won't worry about whether 0 and 12 are {in|out} of the domain.)

b) Suppose that $g(x)$ is the sum of twice the area of the square which has smaller area plus the area of the square which has larger area. Describe the function $g(x)$ algebraically. **Hint** Read the question *carefully*. The answer will be a piecewise-defined function. A complete answer should give all relevant information. **Answer** Here the left square is smaller when $x<6$ and the right square is smaller when $x>6$ (yes, I am ignoring the case $x=6$). Therefore $g(x)=\begin{cases} 2x^2+(12-x)^2 & \text{if } 0 \le x < 6 \\ x^2+2(12-x)^2 & \text{if } 6 < x \le 12 \end{cases}$. When $x=6$, the squares are the same size, and the neatest way to define $g$ (for continuity!) is by $g(6)=3(6^2)$.

**Note** To the right is a graph of both $f(x)$ and $g(x)$ on the interval $[0,12]$. $f(x)$ is the parabola opening up, and $g(x)$ is the weird "curve" with a corner.