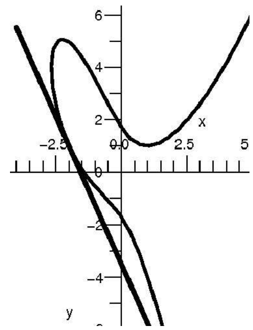
640:151:04–06 **Answers to the Second Exam** **11/24/2006**

Here are answers that would earn full credit. Other methods may also be valid.

(12) 1. In this problem, $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2$. Find the absolute minimum and absolute maximum values of f in the interval $[-2, 1]$. **Answer** $f'(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x-2)(x+1)$, so that $f'(x) = 0$ at 0, 2, and -1. The relevant c.p.'s are 0 and -1. The extreme values must occur there or at the endpoints, so we check four values of f: $f(-2) = \frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - (-2)^2 = 4 + \frac{8}{3} - 4 = \frac{8}{3}$; $f(-1) = \frac{1}{4}(-1)^4 - \frac{1}{3}(-1)^3 - (-1)^2 = \frac{1}{4} + \frac{1}{3} - 1 = -\frac{5}{12}$; $f(0) = 0$; $f(1) = \frac{1}{4}1^4 - \frac{1}{3}1^3 - 1^2 = \frac{1}{4} - \frac{1}{3} - 1 = -\frac{13}{12}$. The absolute maximum value is $\frac{8}{3}$. The absolute minimum value is $-\frac{13}{12}$.

(12) 2. The program `Maple` displays the image shown to the right when asked to graph the equation $y^2 = x^3 - 3xy + 3$.



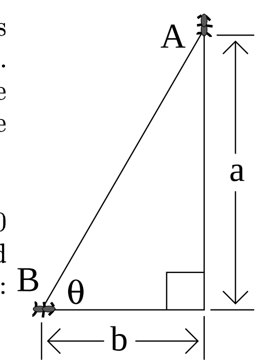
a) Verify by substitution that the point $P = (-2, 1)$ is on the graph of the equation. **Answer** $1^2 = 1$ and $(-2)^3 - 3(-2)1 + 3 = -8 + 6 + 3 = 1$, so the equation is correct.

b) Find $\frac{dy}{dx}$ in terms of y and x. **Answer** d/dx the equation: $2y\frac{dy}{dx} = 3x^2 - 3y - 3x\frac{dy}{dx}$ so that $(2y + 3x)\frac{dy}{dx} = 3x^2 - 3y$ and $\frac{dy}{dx} = \frac{3x^2 - 3y}{2y + 3x}$.

c) Find an equation for the line tangent to the graph at the point $P = (-2, 1)$. **Answer** If $x = -2$ and $y = 1$, $\frac{3x^2-3y}{2y+3x}$ becomes $\frac{9}{-4} = -\frac{9}{4}$. One equation for the line is therefore $y - 1 = -\frac{9}{4}(x + 2)$.

d) Sketch this tangent line in the appropriate place on the image displayed. (Part of the picture is shown.)

(12) 3. Ant **A** is crawling up a vertical pole at .3 meters/minute. At the same time ant **B** is crawling away from the base of the pole on the horizontal ground at .4 meters/minute.



a) If θ is the angle that ant **B** sees between the base of the pole and ant **A**, if a is the distance from ant **A** to the base of the pole, and if b is the distance from ant **B** to the base of the pole, then write a formula for θ as a function of a and b.
Answer Certainly $\tan\theta = \frac{a}{b}$ so $\theta = \arctan\left(\frac{a}{b}\right)$.

b) Use the information provided to compute θ and $\frac{d\theta}{dt}$ at the instant that ant **A** is 10 meters up the pole and ant **B** is 5 meters from the base of the pole. You do **not** need to "simplify" your answer! **Answer** $\theta = \arctan\left(\frac{10}{5}\right)$. Differentiate the answer to a): $\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{a}{b}\right)^2} \cdot \frac{\frac{da}{dt}\cdot b - a\cdot\frac{db}{dt}}{b^2}$. At "the instant", this is $\frac{1}{1+\left(\frac{10}{5}\right)^2} \cdot \frac{.3\cdot 5 - 10\cdot .4}{5^2}$.

(12) 4. Suppose $B(x)$ is a differentiable function with $B(2) = 3$ and that the derivative of B is given by the following formula: $B'(x) = \sqrt{23 - 7x}$. Suppose also that $C(x) = 5x^2 - 3$. Let $A(x) = B(C(x))$.

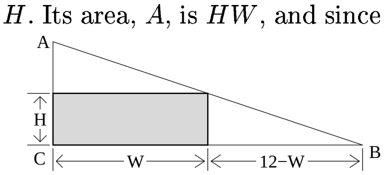
a) Compute $A(1)$. Write a formula for $A'(x)$ only in terms of x and then compute $A'(1)$.
Answer $A(1) = B(C(1)) = B(5-3) = B(2) = 3$ and $A'(x) = B'(C(x))C'(x) = \sqrt{23 - 7(5x^2 - 3)}(10x)$ using the Chain Rule and the information given about B'. Then $A'(1) = \sqrt{23 - 7(5-3)}(10) = 30$.

b) Use your answers to a) and linear approximation to find an approximate value of $A(0.95)$. You do **not** need to "simplify" your answer!**Answer** $A(0.95) \approx A(1) + A'(1)(-.05) = 3 + 30(-.05) = 3 - 1.5 = 1.5$.

c) It is true that $A''(1) = -\frac{260}{3}$. Is the estimate you found in c) likely to be greater than or less than the true value of $A(0.95)$? Give reasoning which supports your answer. **Answer** The graph $y = A(x)$ is likely to be concave down near 1 since the second derivative is negative at $x = 1$. Tangent lines are above such curves, so the linear approximation is likely to be greater than the true value.

Comment The "true value" is about 1.40657. The approximation is 1.5, greater than the true value (but not such a good approximation!). Here's a formula for $A(x)$, which was not requested: $-\frac{2}{21}(44 - 35x^2)^{3/2} + \frac{39}{7}$.

(12) 5. In the right triangle ΔABC, the right angle is at C and the legs are $|AC| = 4$ and $|BC| = 12$. A rectangle is placed inside the triangle, with one corner at C and the opposite corner on the hypotenuse. What are the dimensions and area of the rectangle which has largest area? Briefly explain why you found the rectangle with largest area. **Answer** Suppose the rectangle has width W and height H. Its area, A, is HW, and since ratios of corresponding sides of similar triangles are equal, $\frac{12-W}{H} = \frac{12}{4}$ so that $H = \frac{1}{3}(12 - W)$ and $A = \frac{1}{3}(12 - W)W = 4W - \frac{1}{3}W^2$. Since $\frac{dA}{dW} = 4 - \frac{2}{3}W$, the only critical point is when $W = 6$. Then $H = 2$ and $A = 12$. This value of A is a maximum because $A = 0$ when $W = 0$ and when $W = 12$.



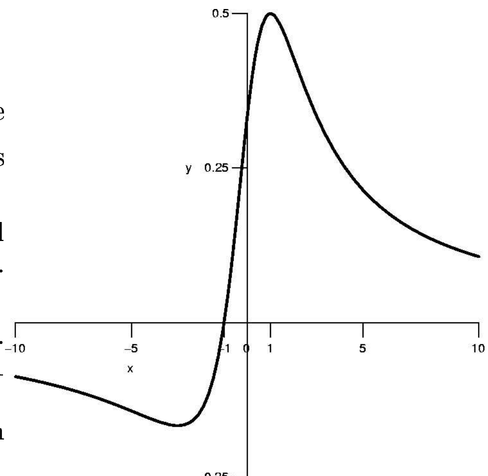
(12) 6. Find the limits.

a) $\lim_{x\to 1} \frac{x^4-4x+3}{(x-1)^2}$. **Answer** Since $1^4-4+3=0$ and $(1-1)^2=0$, we may try l'H: $\lim_{x\to 1} \frac{x^4-4x+3}{(x-1)^2} \overset{\text{l'H}}{=} \lim_{x\to 1} \frac{4x^3-4}{2(x-1)}$. But $4-4=0$ and $2(1-1)=0$ so this is also eligible for l'H: $\lim_{x\to 1} \frac{4x^3-4}{2(x-1)} \overset{\text{l'H}}{=} \lim_{x\to 1} \frac{12x^2}{2} = 6$. The last limit is evaluated by "plugging in" (more officially, the function is continuous at 1, etc.).

b) $\lim_{x\to\infty} (5+3x)^{2/x}$. **Answer** Suppose $W=(5+3x)^{2/x}$. Then $\ln W = \frac{2}{x}\ln(5+3x) = \frac{2\ln(5+3x)}{x}$ Consider $\lim_{x\to\infty} \frac{2\ln(5+3x)}{x}$. As $x\to\infty$, certainly $x\to\infty$, and $5+3x\to\infty$. But ln is increasing and unbounded, so also $2\ln(5+3x)\to\infty$. So we have an indeterminate form of the type $\frac{\infty}{\infty}$, and can try l'H. $\lim_{x\to\infty} \frac{2\ln(5+3x)}{x} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{\frac{2}{5+3x}\cdot 3}{1} = 0$. This is the limit of W as $x\to\infty$, and W was the log of the original function. To get the limit of that function we exponentiate, and $e^0=1$, which is the desired limiting value.

c) $\lim_{x\to\infty} \frac{\frac{\pi}{2}-\arctan x}{e^x-1}$. **Answer** As $x\to\infty$, $\arctan x\to\frac{\pi}{2}$, so the top$\to 0$. As $x\to\infty$, the bottom, e^x-1, goes$\to\infty$. The limit is 0.

(22) 7. In this problem, $f(x)=\frac{x+1}{x^2+3}$.



a) What are $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$? **Answer** Both limits are indeterminate ($\frac{\infty}{\infty}$) and l'H allows us to consider the limit of $\frac{1}{2x}$ as $x\to\pm\infty$. These are both 0: $\lim_{x\to+\infty} f(x)=0$ and $\lim_{x\to-\infty} f(x)=0$.

b) Compute $f'(x)$ *carefully*, since the result is needed for successful completion of the remainder of the problem. Simplify your result. **Answer** $f'(x)=\frac{-x^2-2x+3}{(x^2+3)^2}$

c) Find all solutions of $f'(x)=0$. For each such x, compute $f(x)$. **Answer** $-x^2-2x+3=-(x-1)(x+3)$ so solutions are $x=1$ and $x=-3$. Then $f(-3)=-\frac{1}{6}$ and $f(1)=\frac{1}{2}$.

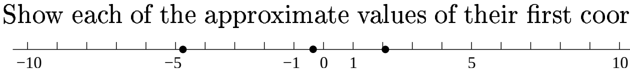
d) Where is $f'(x)>0$? Where is $f'(x)<0$? **Answer** The bottom of f' is always positive, so the sign is determined by the top. The sign changes at -3 and 1. $f'(x)>0$ when $-3<x<1$. $f'(x)<0$ when $x<-3$ and when $1<x$.

e) Sketch a graph of $y=f(x)$. The conclusions of parts a) and c) and d) should all be used here. The scales of the vertical and horizontal axes are very different. **Answer** A `Maple` graph is shown above.

f_1) How many solutions does the equation $f(x)=.07$ have? (You are *not* asked to find the solutions!) **Answer** $f(x)=.07$ has 2 solutions.

f_2) How many solutions does the equation $f(x)=.87$ have? (You are *not* asked to find the solutions!) **Answer** $f(x)=.87$ has 0 solutions.

g) What is the *range* of f? (That is, the collection of all y's for which $f(x)=y$ has at least one solution.) **Answer** The range of f is $\left[-\frac{1}{6},\frac{1}{2}\right]$.

h) You do <u>not</u> need to compute $f''(x)$ to answer the following question: how many inflection points must $y=f(x)$ have, and what can you say about the approximate location of these inflection points?
Answer The number of inflection points is 3. Show each of the approximate values of their first coordinates on the number line below with a •. **Answer** 

Comment The actual numbers are (about) -4.75877, $-.30541$, and 2.0642.

(6) 8. Suppose that $f(x)$ is a differentiable function, and that for all x, $4<f'(x)<6$. Suppose also that $f(0)=-2$.

a) Explain why $f(5)$ must be positive. You should quote a specific result from this course and explain its relevance. **Answer** Since f is <u>differentiable</u>, the <u>Mean Value Theorem</u> applies: $\frac{f(5)-f(0)}{5-0}=f'(x)$ for some x between 0 and 5, so $f(5)=f(0)+f'(x)\cdot 5 \geq -2+4\cdot 5=18$.

b) Explain why $f(x)=0$ must have a solution in the interval $[0,5]$. You may assume the sign information stated in a) here. You should quote a specific result from this course and explain its relevance.
Answer Since f is differentiable, it is <u>continuous</u> on the interval $[0,5]$, and the <u>Intermediate Value Theorem</u> applies. Since $f(0)<0$ and $f(5)>0$, the Theorem states that there must be some number w in the interval <u>$[0,5]$</u> so that $f(w)=0$.