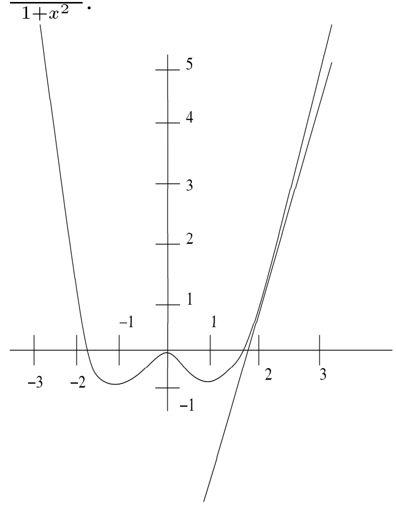
Some answers to sample problems for the first exam in Math 151, sections 4, 5, and 6

S. Blight

October 12, 2006

The following solutions are meant to help you understand the material and so possibly show slightly more detail than would be required on the exam.

1) Suppose $f(x) = \frac{x^4-3x^2}{1+x^2}$.



a) Compute $f'(x)$.

Using the quotient rule,

$$f'(x) = \frac{(1+x^2)(4x^3-6x)-(x^4-3x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{4x^3-6x+4x^5-6x^3-2x^5+6x^3}{(1+x^2)^2}$$

$$= \frac{2x^5+4x^3-6x}{(1+x^2)^2}$$

b) Find an equation for the line tangent to $y = \frac{x^4-3x^2}{1+x^2}$ when $x = 2$.

First, note that the slope of the tangent line at $x = 2$ is the derivative $f'(2)$. By (a), $f'(2) = \frac{64+32-12}{25} = \frac{84}{25}$. Then, the equation for the tangent line is

$y = \frac{84}{25}x + b$ for some constant b. We also know that the tangent line must go through the point $(2, f(2)) = (2, \frac{4}{5})$. So, $\frac{4}{5} = \frac{84}{25}(2) + b$, and so $b = \frac{-148}{25}$. In conclusion the equation of the tangent line to $f(x)$ at $x = 2$ is

$$y = \frac{84}{25}x - \frac{148}{25} = \frac{1}{25}(84x - 148).$$

c) To the right is a graph of $y = \frac{x^4-3x^2}{1+x^2}$. Sketch the line whose equation you have found in (a) on this graph.

d) For which values of x is the line tangent to $y = \frac{x^4-3x^2}{1+x^2}$ horizontal?

A horizontal tangent line occurs when the slope of the tangent line is zero, which is when the derivative is zero. Therefore, we want to find the x values such that $f'(x) = 0$. That is, $\frac{2x^5+4x^3-6x}{(1+x^2)^2}$. This occurs when $0 = 2x^5 + 4x^3 - 6x = 2x(x^4 + 2x^2 - 3) = 2x(x^2 + 3)(x^2 - 1)$. Note that $x^2 + 3 \neq 0$ for all real x, so $f'(x) = 0$ if $2x = 0$ or $x^2 - 1 = 0$. So, the solutions are $x = 0$, $x = 1$, and $x = -1$. These solutions agree with the picture.

2) Suppose $f(x) = x^2 + 2 + 3\sin x$, and that g is a differentiable function about which the following is known: $g(0) = 5$ and $g'(0) = -2$. Compute the following quantities (an answer alone will *not* receive full credit): $(f + g)'(0)$; $(f \cdot g)'(0)$; $\left(\frac{f}{g}\right)'(0)$.

By the addition rule, $(f+g)'(x) = f'(x)+g'(x)$, so $(f+g)'(0) = f'(0)+g'(0)$. We need to calculate $f'(x)$, so $f'(x) = 2x + 3\cos x$ and $f'(0) = 3$ and $(f + g)'(0) = 3 + -2 = 1$.

By the product rule, $(f\cdot g)'(x) = f(x)g'(x)+f'(x)g(x)$, so $(f\cdot g)'(0) = f(0)g'(0)+ f'(0)g(0)$. We also know $f(0) = 2$, so $(f \cdot g)'(0) = 2(-2) + 3(5) = -4 + 15 = 11$.

By the quotient rule $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$, so $\left(\frac{f}{g}\right)'(0) = \frac{g(0)f'(0)-f(0)g'(0)}{g(0)^2} = \frac{5(3)-2(-2)}{5^2} = \frac{19}{25}$

3) Write the definition of derivative as a limit and *use this definition* to find the derivative of $f(x) = \frac{1}{\sqrt{x}}$.

The definition of the derivative is that if $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and equals L, then $f'(x) = L$.

First note that $f(x)$ is only defined on $[0, \infty)$. Therefore, $f(x)$ is not differentiable for $x \leq 0$. So, consider $x > 0$. Then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \lim_{h\to 0} \frac{h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \frac{1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} = \frac{1}{2x\sqrt{x}}$$

4) Suppose $f(x) = \frac{3}{x^2} - x^2 + e^{-7x}$. Explain why $f(x) = 0$ must have a solution in the interval $[1, 2]$. Your answer should use complete English sentences and quote a specific result from this course, explaining the relevance of this result.
First, note that $f(1) = 3 - 1 + e^{-7} = 2 + e^{-7} > 0$ since $e^{-7} > 0$. Also, $f(2) = \frac{3}{4} - 4 + e^{-14} = \frac{-13}{4} + e^{-14} < \frac{-13}{4} + 1 = \frac{-9}{4} < 0$, because $e^{-14} < 1$. Therefore, $f(2) < 0 < f(1)$. In addition, $f(x)$ is continuous on the interval $[1, 2]$. Thus, by the Intermediate Value Theorem, there exists a number c in $(1, 2)$ such that $f(c) = 0$. Then, $f(x) = 0$ has a solution in the interval $[1, 2]$.

5) Suppose that $f(x) = \frac{|x^2-1|}{x-1}$.
a) Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.
When calculating the right-hand limit, we have $x > 1$, so $x^2 > 1$, and $x^2 - 1 > 0$, so $|x^2 - 1| = x^2 - 1$.

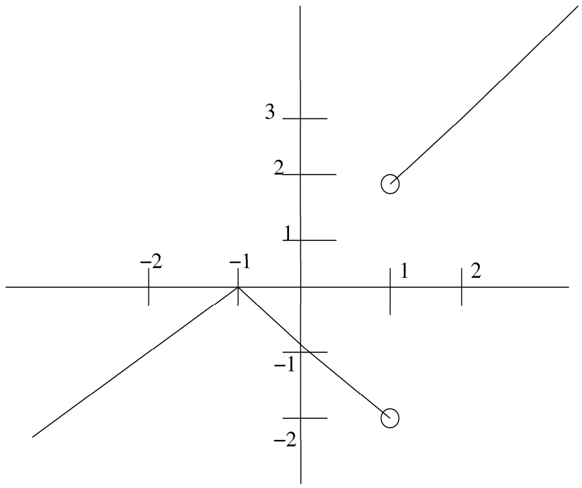
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{|x^2-1|}{x-1} = \lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1^+} x + 1 = 2$$

When calculating the left-hand limit, $x < 1$. We can also assume that $x > 0$ since we are approaching 1. Then $0 < x < 1$, so $x^2 < 1$ and $x^2 - 1 < 0$, so $|x^2 - 1| = -(x^2 - 1)$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{|x^2-1|}{x-1} = \lim_{x\to 1^-} \frac{-(x^2-1)}{x-1} = \lim_{x\to 1^-} \frac{-(x+1)(x-1)}{x-1}$$
$$= \lim_{x\to 1^-} -(x+1) = -2$$

b) Does $\lim_{x\to 1} f(x)$ exist? We have $\lim_{x\to 1^+} f(x) = 2 \neq -2 = \lim_{x\to 1^-} f(x)$. Therefore, the right-hand limit does not equal the left-hand limit, so $\lim_{x\to 1} f(x)$ does not exist.

c) Sketch a graph of $y = f(x)$ for x between -3 and 3. For $x > 1$, $f(x) = x + 1$. For $-1 < x < 1$, $x^2 < 1$ so $x^2 - 1 < 0$ and $f(x) = -(x + 1)$. For $x \le -1$, $x^2 \ge 1$, so $x^2 - 1 \ge 0$, so $f(x) = x + 1$. The function is not defined at $x = 1$.



6)a) If $f(x) = \frac{6e^x}{x^3-7}$, what is $f'(x)$? Please do not "simplify" your answer.
By the quotient rule,

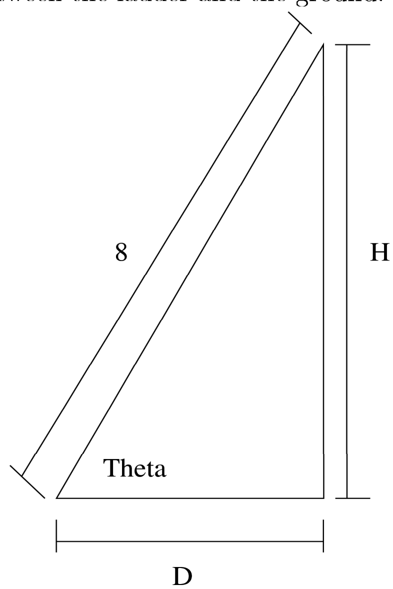
$$f'(x) = \frac{(x^3-7)(6e^x) - (6e^x)(3x^2)}{(x^3-7)^2}$$

b) If $f(x) = (x^7 + \cos x)(4x^4 + 3x^3)$, what is $f'(x)$? Please do not "simplify" your answer.

By the product rule,

$$f'(x) = (x^7 + \cos x)(16x^3 + 9x^2) + (7x^6 - \sin x)(4x^4 + 3x^3)$$

7) A ladder which is 8 feet long has one end on flat ground and the other end on the vertical wall of a building. H is the height from the ground to the point at which the ladder touches the building, and D is the distance between the bottom of the ladder and the bottom of the wall. θ is the acute angle between the ladder and the ground.



a) Write H as a function of D. That is, give a formula for H involving D and no other variable. What is the domain of this function when used to describe this problem? (The answer should be related to the problem's geometry.)

By the Pythagorean theorem, $D^2 + H^2 = 8^2 = 64$, so $H^2 = 64 - D^2$ and $H = \pm\sqrt{64 - D^2}$, since $H \geq 0$, $H = \sqrt{64 - D^2}$. The domain for the height function is $[0, 8]$, where 0 corresponds to the ladder being straight up against the wall and 8 corresponds to the ladder lying on the ground.

b) Write H as a function of θ. That is, give a formula for H involving θ and no other variable. What is the domain of this function when used to describe this problem? (The answer should be related to the problem's geometry.)

By trigonometry, $\sin\theta = \frac{H}{8}$, so $H = 8\sin\theta$. The domain for the height function is $[0, \frac{\pi}{2}]$, where $\theta = 0$ corresponds to the ladder lying on the ground and $\theta = \frac{\pi}{2}$ corresponds to the ladder being flat against the wall.

8) Find, as precisely as possible (you may need to use values of ln), equations for all horizontal and vertical asymptotes of $f(x) = \frac{5e^x+4e^{-x}}{3e^{2x}-2e^{-x}}$.

To find any possible horizontal asymptotes, we want to look at the limit of $f(x)$ as x goes to infinity and the limit as x goes to $-\infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{5e^x + 4e^{-x}}{3e^{2x} - 2e^{-x}} = \lim_{x\to\infty} \frac{e^{2x}(5e^{-x} + 4e^{-3x})}{e^{2x}(3 - 2e^{-3x})} = \lim_{x\to\infty} \frac{5e^{-x} + 4e^{-3x}}{3 - 2e^{-3x}}$$

Then, $\lim_{x\to\infty} 5e^{-x} = \lim_{x\to\infty} 4e^{-3x} = \lim_{x\to\infty} -2e^{-3x} = 0$, so $\lim_{x\to\infty} 5e^{-x} + 4e^{-3x} = 0$ and $\lim_{x\to\infty} 3 - 2e^{-3x} = 3$, so $\lim_{x\to\infty} f(x) = 0$ and $y = 0$ is a horizontal asymptote.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{5e^x + 4e^{-x}}{3e^{2x} - 2e^{-x}} = \lim_{x\to-\infty} \frac{e^{-x}(5e^{2x} + 4)}{e^{-x}(3e^{3x} - 2)} = \lim_{x\to-\infty} \frac{5e^{2x} + 4}{3e^{3x} - 2}$$

Then, $\lim_{x\to-\infty} 5e^{2x} = 0 = \lim_{x\to-\infty} 3e^{3x}$, so $\lim_{x\to-\infty} 5e^{2x} + 4 = 4$ and $\lim_{x\to-\infty} 3e^{3x} - 2 = -2$, so
$\lim_{x\to-\infty} f(x) = \frac{4}{-2} = -2$, so $y = -2$ is a horizontal asymptote.

A line $x = a$ is a vertical asymptote if any of the limits as x approaches a is $\pm\infty$. This occurs if the denominator approaches zero and the numerator does not. We now want to find when $3e^{2x} - 2e^{-x}$ approaches zero. This occurs when $3e^{2x} - 2e^{-x} = 0$, so $3e^{2x} = 2e^{-x}$, and $e^{3x} = \frac{2}{3}$, so $x = \frac{1}{3}\ln\left(\frac{2}{3}\right)$. $5e^x + 4e^{-x} > 0$ for all x and is continuous, so $\lim_{x\to\ln(2/3)/3} 5e^x + 4e^{-x} > 0$. Thus, $x = \ln(2/3)/3$ is a vertical asymptote of $f(x)$.

9) Below is a graph of the function $y = f(x)$.

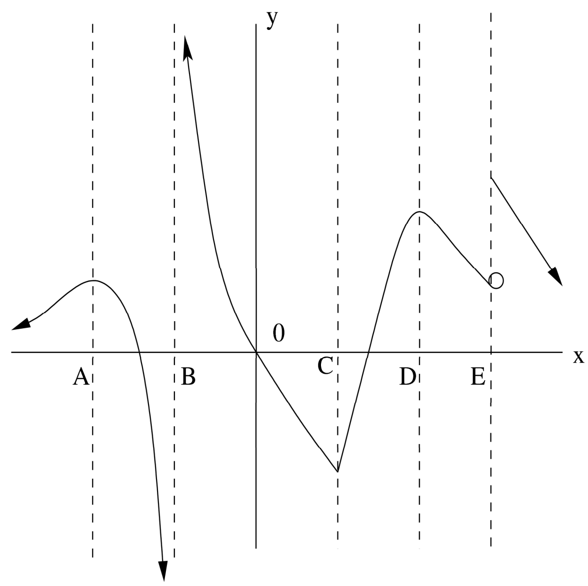
a) Use the axes below (make a bigger vertical axis, please!) to sketch a graph of $y = f'(x)$ as well as you can.

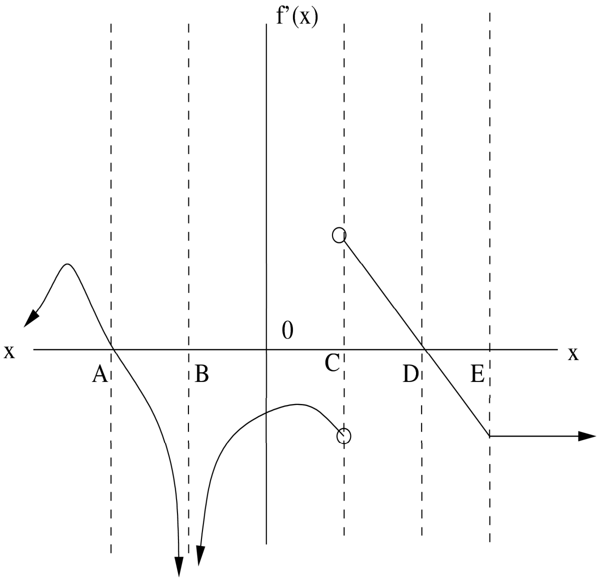
b) For what $x's$ in the list $\{A, B, 0, C, D, E\}$ is $f(x)$ continuous.

For the points $A, 0, C$ and D, the limits of the function equal the function value, so f is continuous at these points. At the point B, the function is not defined, so f is not continuous at B. At the point $x = E$, the right-hand and left-hand limits are not equal so the function is not continuous at E.

c) For which x's in the list $\{A, B, 0, C, D, E\}$ is $f(x)$ differentiable?

$f(x)$ is differentiable at $A, 0$ and D because the curve seems to be smooth and has a nice tangent line (it is locally linear under sufficient magnification) at these points. $f(x)$ is not differentiable at B or E because f is not continuous at B and E. $f(x)$ is not differentiable at C because there is a sharp corner at C.


y
0
x
A
B
C
D
E


f'(x)
0
x
A
B
C
D
E