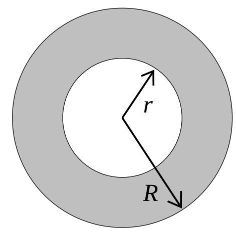
Math 151:4–6 November 13, 2006

Review material for the second exam for Math 151

Background information

The second exam will cover the contents of lectures 10 through 19 of the course, *but* due to the cumulative nature of the material, knowledge of what was done earlier in the course is essential to success on this exam. Some techniques not tested on the first exam (the chain rule, implicit differentiation) will appear on this exam. Other than that, the emphasis will be on the uses of the derivative: linear approximation, Newton's method, L'Hôpital's Rule, the Mean Value Theorem (MVT), relationships between a function and its first and second derivatives, and uses of derivatives in simple models of real problems (related rates, max/min). No notes or calculators (or other electronics!) may be used but a formula sheet written by the course coordinator will be handed out with the exam. A link to this formula sheet is on the course web page. The coordinator of the course has prepared sample problems and answers which are linked to the course web page. These are useful for review (the course coordinator will be the primary author of the final exam), as are the assigned textbook homework problems. Below are sample problems written by the lecturer who will write the second exam. Again, an answer alone may not receive full credit. Supporting work will almost always be needed to earn full credit.

Sample problems



0. Two circles have the same center. The inner circle has radius r which is increasing at the rate of 3 inches per second. The outer circle has radius R which is increasing at the rate of 2 inches per second. Suppose that A is the area of the region *between* the circles. At a certain time, r is 7 inches and R is 10 inches. What is A at that time? How fast is A changing at that time? Is A increasing or decreasing at that time?

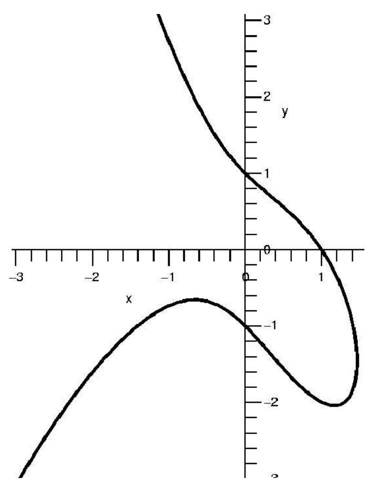
1. In this problem, the <u>derivative</u> of $f(x)$, $f'(x)$, is given by the formula $f'(x) = (x^2 - 4)e^{(x^2)}$.

a) What are the critical points of f? What types of critical points (rel max, rel min, neither) does f have? In what intervals does f increase? Decrease?

b) Compute $f''(x)$ carefully and "simplify".

c) In what intervals is $y = f(x)$ concave up? Concave down? Does $f(x)$ have any inflection points? If your answer is yes, where are they?

d) Use the information obtained in a) and c) to sketch a qualitatively correct graph of $y = f(x)$. Label any inflection points with **I** and any relative maxima with **M** and any relative minima with **m**.*



2. The program `Maple` displays the image shown to the right when asked to graph the equation $\diamondsuit$: $x^3 + 2xy + y^2 = 1$.

a) Verify by substitution that the point $P = (1, -2)$ is on the graph of the equation.

b) Differentiate the equation $\diamondsuit$ with respect to x.

c) Substitute $x = 1$ and $y = -2$ into the result of b) and use the information to write an equation for the tangent line to $\diamondsuit$ at P.

d) Near P the curve shown seems to be concave up. Differentiate the result of b) with respect to x again and verify that the value of y'' at P has a suitable sign. (What should the sign be?)

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* Please don't try to find a formula for $f(x)$. Such a formula involves functions which you're not likely to know right now, and the actual formula is not likely to help with the sketch.

3. Suppose $f(x)$ is a differentiable function with $f(8) = 5$, $f'(8) = 3$ and $f''(8) = -2$. If $F(x) = f\left(x^3\right)$, compute $F(2)$, $F'(2)$, and $F''(2)$.

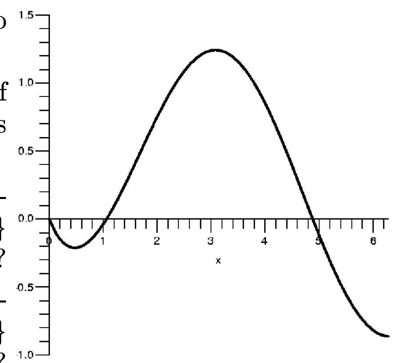
4. Find the limit, which could be a specific real number or $+\infty$ or $-\infty$. In each case, briefly indicate your reasoning, based on calculus or algebra or properties of functions.

a) $\lim_{x\to\infty} \frac{\ln(\ln(x))}{(\ln(x))^2}$ b) $\lim_{x\to 3} \frac{x^2-5x+6}{x+3}$ c) $\lim_{x\to 0} \frac{\sin(x)-x}{(e^x-1)^2}$ d) $\lim_{x\to\infty} \left(\arctan(x^2) - \arctan(x)\right)$

5. a) Suppose $f(x) = x^3 + 3x + \sin(x-5) + 2$. Explain carefully why the statement "If $0 < x < 1$, then $f(0) < f(x) < f(1)$" is correct. You may need a result from the course. Explain why it is applicable. Give simple estimates for $f(0)$ and $f(1)$.

b) Now suppose that $F'(x) = f(x)$, with $f(x)$ as defined in a). Use the results of a) to answer the following question: if you know that $F(0) = -3$, how big and how small can $F(1)$ be? Explain your reasoning and your computation briefly.

6. In this problem, $f(x) = \frac{1}{1+x} - \cos x$. A graph of $y = f(x)$ is shown to the right (the vertical and horizontal axes have different scales).

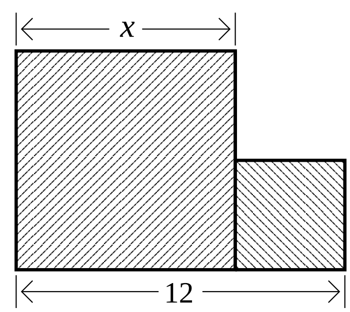


a) Write an expression showing how x_n, an approximation for a root of $f(x) = 0$, is changed to an improved approximation, x_{n+1}, using Newton's method.

b) Suppose $A_0 = 2$. Draw on the graph (with labels!) the next approximations A_1 and A_2 obtained using Newton's method. Does the sequence $\{A_n\}$ of approximations converge to the smallest *positive* root of $\frac{1}{1+x} - \cos x = 0$?

c) Suppose $B_0 = 4$. Draw on the graph (with labels!) the next approximations B_1 and B_2 obtained using Newton's method. Does the sequence $\{B_n\}$ of approximations converge to the smallest *positive* root of $\frac{1}{1+x} - \cos x = 0$?

7. Two squares are placed so their sides are touching, as shown. The sum of the lengths of one side of each square is 12 feet. Suppose the length of a side of the left square is x feet. The left square is painted with paint costing \$5 per square foot. The right square is painted with paint costing \$7 per square foot.



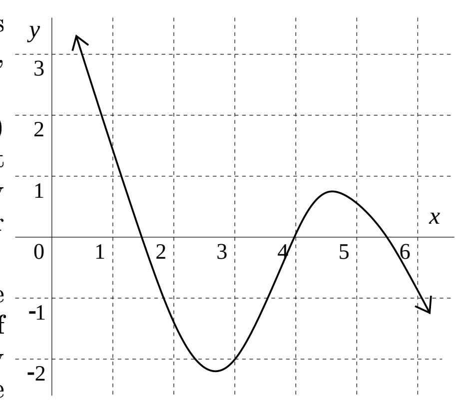
a) Find $C(x)$, the cost of painting both squares, in terms of x.

b) For which x will the cost be a minimum? What *is* the minimum cost? Justify your answer with calculus.

8. The graph of $y = f'(x)$, the *derivative* of the function $f(x)$, is shown to the right. (Both parts of this problem, otherwise unrelated, use information from the graph of the derivative of $f(x)$.)

a) Use information from the graph of $f'(x)$ to find (as well as possible) the x where the *maximum value* of $f(x)$ in the interval $1 \le x \le 3$ must occur. Explain using calculus why your answer is correct (explain why the value of $f(x)$ you select is larger than $f(x)$ at *any* other number in the interval.)

b) Suppose that $f(3) = 5$. Use information from the graph and the tangent line approximation for $f(x)$ to find an approximate value of $f(3.04)$. Explain using calculus and information from the graph why your approximate value for $f(3.04)$ is greater than or less than the exact value of $f(3.04)$.



Graph of $f'(x)$, the *derivative* of $f(x)$