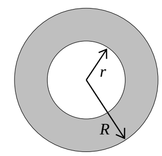
Some answers to review questions for the second exam for Math 151

0. Two circles have the same center. The inner circle has radius r which is increasing at the rate of 3 inches per second. The outer circle has radius R which is increasing at the rate of 2 inches per second. Suppose that A is the area of the region *between* the circles. At a certain time, r is 7 inches and R is 10 inches. What is A at that time? How fast is A changing at that time? Is A increasing or decreasing at that time?



Answer $A = \pi R^2 - \pi r^2$. If $R = 10$ and $r = 7$, $A = \pi(10^2) - \pi(7^2) = 51\pi$. If we d/dt the equation, the result is $A' = 2\pi R\frac{dR}{dt} - 2\pi r\frac{dr}{dt} = 2\pi(10)(2) - 2\pi(7)(3) = -2\pi$. The area is changing at 2π inches per second and it is *decreasing*.

1. In this problem, the <u>derivative</u> of $f(x)$, $f'(x)$, is given by the formula $f'(x) = (x^2-4)e^{(x^2)}$.

a) What are the critical points of f? What types of critical points (rel max, rel min, neither) does f have? In what intervals does f increase? Decrease?

Answer First, we note that the critical points of f occur when $f'(x) = 0$ or $f'(x)$ does not exist. Since $f'(x)$ is defined for all real numbers, $f'(x)$ exists everywhere and we look to see when $0 = f'(x) = (x-2)(x+2)e^{(x^2)}$, so our critical points are $x = 2$ and $x = -2$. Next, notice that for $x < -2$, $f'(x) > 0$, for $-2 < x < 2$, $f'(x) < 0$, and for $x > 2$, $f'(x) > 0$. Therefore, $x = -2$ is a relative maximum and $x = 2$ is a relative minimum.

b) Compute $f''(x)$ carefully and "simplify".

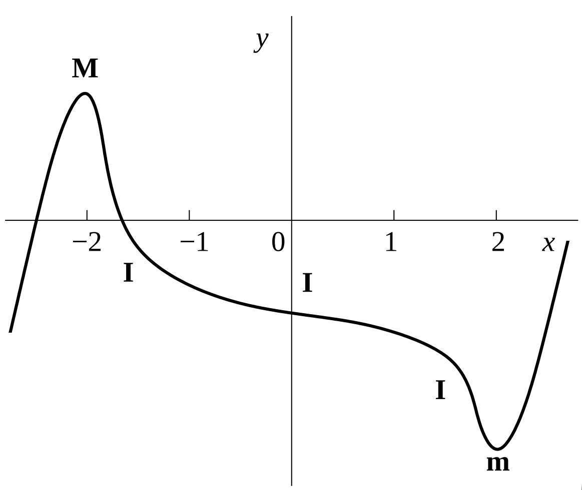
Answer $f''(x) = (x^2-4)e^{(x^2)}(2x) + e^{(x^2)}(2x) = 2e^{(x^2)}(x)(x^2-4+1) = 2e^{(x^2)}(x-\sqrt{3})(x+\sqrt{3})x$.

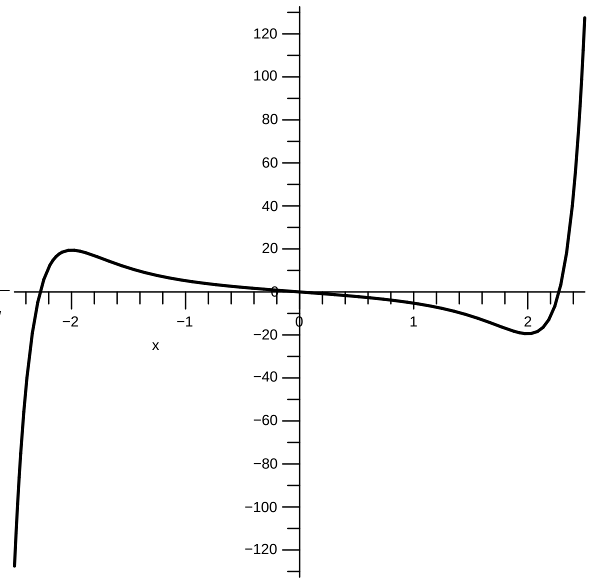
c) In what intervals is $y = f(x)$ concave up? Concave down? Does $f(x)$ have any inflection points? If your answer is yes, where are they?

Answer For $x < -\sqrt{3}$, $f''(x) < 0$. For $-\sqrt{3} < x < 0$, $f''(x) > 0$. For $0 < x < \sqrt{3}$, $f''(x) < 0$. For $x > \sqrt{3}$, $f''(x) > 0$. Therefore, $f(x)$ is concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ and $f(x)$ is concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$. Then, there are points of inflection at $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$.

d) Use the information obtained in a) and c) to sketch a qualitatively correct graph of $y = f(x)$. Label any inflection points with **I** and any relative maxima with **M** and any relative minima with **m**.*

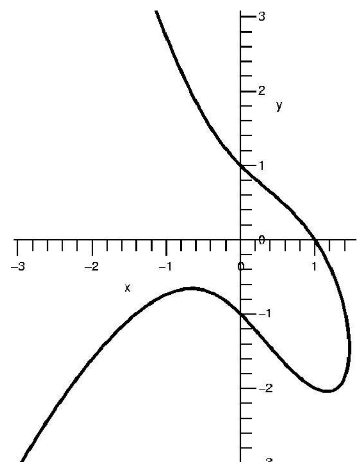
Answer My "hand-drawn" (?) answer to this is displayed to the left. I also used `Maple` and got the graph of one $f(x)$ which is shown on the right (there are infinitely many correct $f(x)$'s, separated by $+C$'s, so geometrically these are just up-and-down translates of each other). Because of the exponential factor in the formula for $f'(x)$, the vertical axis has a very different scale from the horizontal axis.





* Please don't try to find a formula for $f(x)$. Such a formula involves functions which you're not likely to know right now, and the actual formula is not likely to help with the sketch.

2. The program `Maple` displays the image shown to the right when asked to graph the equation $\diamondsuit$: $x^3 + 2xy + y^2 = 1$.



a) Verify by substitution that the point $P = (1, -2)$ is on the graph of the equation.
Answer $1^3 + 2(1)(-2) + (-2)^2 = 1 - 4 + 4 = 1$, so P is on the graph of the equation.

b) Differentiate the equation $\diamondsuit$ with respect to x.
Answer $3x^2 + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$, so $(2x + 2y)\frac{dy}{dx} = -3x^2 - 2y$ and finally $\frac{dy}{dx} = \frac{-3x^2 - 2y}{2x+2y}$.

c) Substitute $x = 1$ and $y = -2$ into the result of b) and use the information to write an equation for the tangent line to $\diamondsuit$ at P.
Answer $\frac{dy}{dx}(P) = \frac{-3+4}{2-4} = -\frac{1}{2}$. So the equation of the tangent line is given by $y - (-2) = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x - \frac{3}{2}$.

d) Near P the curve shown seems to be concave up. Differentiate the result of b) with respect to x again and verify that the value of y'' at P has a suitable sign. (What should the sign be?)
Answer We differentiate the answer to b):

$$y'' = \frac{(2x + 2y)(-6x - 2\frac{dy}{dx}) - (-3x^2 - 2y)(2 + 2\frac{dy}{dx})}{(2x + 2y)^2}$$

and therefore $y''(P) = \frac{(2-4)(-6+1)-(-3+4)(2-1)}{(2-4)^2} = \frac{10-1}{4} = \frac{9}{4} > 0$. "Concave up" should correspond to *positive* second derivative.

3. Suppose $f(x)$ is a differentiable function with $f(8) = 5$, $f'(8) = 3$ and $f''(8) = -2$. If $F(x) = f(x^3)$, compute $F(2)$, $F'(2)$, and $F''(2)$.
Answer $F(2) = f(2^3) = f(8) = 5$; $F'(x) = f'(x^3)(3x^2)$, so $F'(2) = f'(8)(12) = 36$; $F''(x) = f'(x^3)(6x) + (3x^2)(f''(x^3))(3x^2)$, so $F''(2) = f'(8)(12) + (9 \cdot 2^4)f''(8) = 36 + 9 \cdot 16 \cdot (-2) = -252$.

4. Find the limit, which could be a specific real number or $+\infty$ or $-\infty$. In each case, briefly indicate your reasoning, based on calculus or algebra or properties of functions.

a) $\lim\limits_{x\to\infty} \frac{\ln(\ln(x))}{(\ln(x))^2}$
Answer We know that $\lim\limits_{x\to\infty} \ln(\ln(x)) = \infty$ and $\lim\limits_{x\to\infty} (\ln(x))^2 = \infty$, so we can apply l'Hôpital's rule.

$$\lim_{x\to\infty} \frac{\ln(\ln(x))}{(\ln(x))^2} = \lim_{x\to\infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{2(\ln(x))\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{2(\ln(x))^2} = 0$$

b) $\lim\limits_{x\to 3} \frac{x^2-5x+6}{x+3}$
Answer $\lim\limits_{x\to 3} \frac{x^2-5x+6}{x+3} = \frac{3^2-15+6}{6} = 0$ since $\frac{x^2-5x+6}{x+3}$ is continuous at $x = 3$.

c) $\lim\limits_{x\to 0} \frac{\sin(x)-x}{(e^x-1)^2}$
Answer $\lim\limits_{x\to 0} \sin(x) - x = 0 = \lim\limits_{x\to 0} (e^x - 1)^2$ so we can apply l'Hôpital's rule.

$$\lim_{x\to 0} \frac{\sin(x) - x}{(e^x - 1)^2} = \lim_{x\to 0} \frac{\cos x - 1}{2(e^x - 1)(e^x)}$$

But $\lim\limits_{x\to 0} \cos(x) - 1 = 0 = \lim\limits_{x\to 0} 2(e^x - 1)e^x$, so we can apply l'Hôpital's rule again.

$$\lim_{x\to 0} \frac{\sin x - x}{(e^x - 1)^2} = \lim_{x\to 0} \frac{-\sin(x)}{2(e^x - 1)e^x + e^x(2e^x)} = \frac{0}{2} = 0$$

since $\frac{-\sin(x)}{2(e^x-1)e^x+e^x(2e^x)}$ is continuous at $x = 0$.

d) $\lim\limits_{x\to\infty} \left(\arctan(x^2) - \arctan(x)\right)$

Answer $\lim_{x\to\infty} \arctan(x^2) = \dfrac{\pi}{2}$ and $\lim_{x\to\infty} \arctan(x) = \dfrac{\pi}{2}$, so $\lim_{x\to\infty} \arctan x^2 - \arctan x = \frac{\pi}{2} - \frac{\pi}{2} = 0$.

5. a) Suppose $f(x) = x^3 + 3x + \sin(x-5) + 2$. Explain carefully why the statement "If $0 < x < 1$, then $f(0) < f(x) < f(1)$" is correct. You may need a result from the course. Explain why it is applicable. Give simple estimates for $f(0)$ and $f(1)$.

Answer $f'(x) = 3x^2 + 3 + \cos(x-5)$. For $0 < x < 1$, $-1 \le \cos(x-5) \le 1$ and $0 \le 3x^2 \le 3$, so $0 + 3 + -1 \le 3x^2 + 3 + \cos(x-5) \le 3 + 3 + 1$ and $2 \le f'(x) \le 7$ for $0 < x < 1$. In particular, $f'(x) > 0$ for $0 < x < 1$, so f is strictly increasing and $f(0) < f(x) < f(1)$ for $0 < x < 1$. This is the "result from the course": if a differentiable function on an interval has a positive derivative, then the function is increasing on that interval. Also, $f(0) = \sin(-5) + 2$, so $-1 + 2 \le f(0) \le 1 + 2$, so $1 \le f(0) \le 3$. In addition, $f(1) = 1 + 3 + \sin(-4) + 2$, so $5 \le f(1) \le 7$.

b) Now suppose that $F'(x) = f(x)$, with $f(x)$ as defined in a). Use the results of a) to answer the following question: if you know that $F(0) = -3$, how big and how small can $F(1)$ be? Explain your reasoning and your computation briefly.

Answer Since $F(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$, we can apply the Mean Value Theorem. Therefore, there exists c in $(0,1)$ such that $F'(c) = \frac{F(1)-F(0)}{1-0}$. That means $f(c) = F(1) - F(0)$, so $f(c) - 3 = F(1)$. For c in $(0,1)$, $1 \le f(0) < f(c) < f(1) \le 7$, so $-2 < F(1) < 4$.

6. In this problem, $f(x) = \frac{1}{1+x} - \cos x$. A graph of $y = f(x)$ is shown to the right (the vertical and horizontal axes have different scales).

a) Write an expression showing how x_n, an approximation for a root of $f(x) = 0$, is changed to an improved approximation, x_{n+1}, using Newton's method.

Answer The general formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Therefore, in this problem,

$$x_{n+1} = x_n - \frac{\frac{1}{1+x_{n+1}} - \cos x_{n+1}}{\frac{-1}{(x_{n+1}+1)^2} + \sin x_{n+1}}.$$

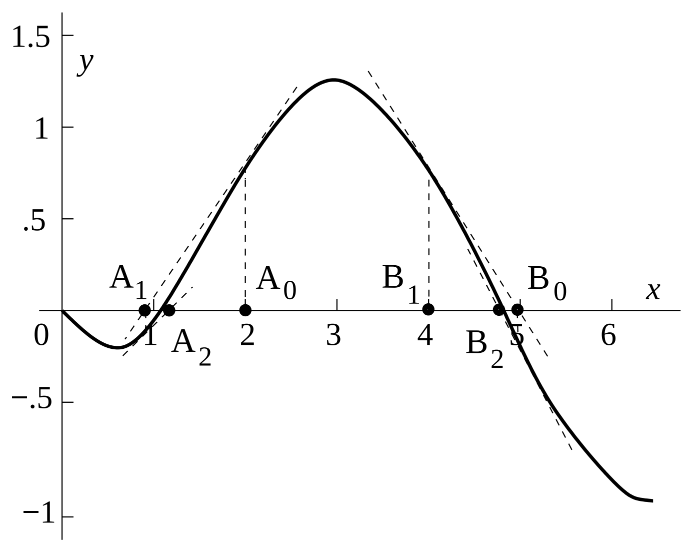
b) Suppose $A_0 = 2$. Draw on the graph (with labels!) the next approximations A_1 and A_2 obtained using Newton's method. Does the sequence $\{A_n\}$ of approximations converge to the smallest *positive* root of $\frac{1}{1+x} - \cos x = 0$?

Answer Yes, the $\{A_n\}$ converge to the smallest positive root, around $x = 1$.

c) Suppose $B_0 = 4$. Draw on the graph (with labels!) the next approximations B_1 and B_2 obtained using Newton's method. Does the sequence $\{B_n\}$ of approximations converge to the smallest *positive* root of $\frac{1}{1+x} - \cos x = 0$?

Answer No, the sequence $\{B_n\}$ converges to the root around $x = 5$.

Comment The dashed lines are intended to show the transitions between the iterations of Newton's method: up to the graph, along the tangent line until it hits the horizontal axis, etc.



Here are some numbers, if you are interested: A_1=1.061021024, A_2=1.065389636, A_3=1.065378990; A_3 "is" the root near 1 to 10 digit accuracy. And B_1=5.071336531, B_2=4.877666749, B_3=4.883191337, B_4=4.883193988; B_4 "is" the root near 5 to 10 digit accuracy.

7. Two squares are placed so their sides are touching, as shown. The sum of the lengths of one side of each square is 12 feet. Suppose the length of a side of the left square is x feet. The left square is painted with paint costing \$5 per square foot. The right square is painted with paint costing \$7 per square foot.

a) Find $C(x)$, the cost of painting both squares, in terms of x.

Answer $C(x) = 5x^2 + 7(12-x)^2$

b) For which x will the cost be a minimum? What *is* the minimum cost? Justify your answer with calculus.

Answer The domain of the cost function, C, in this problem is $[0, 12]$. Since $C'(x) = 10x + 14(12-x)(-1) = 24(x-7)$, we know $C'(x) = 0$ if $x = 7$. We need to know why (or if!) C has a minimum at 7.

0^{th} derivative test

$C(0) = 7 \cdot 12^2$ and $C(12) = 5 \cdot 12^2$ and $C(7) = 5 \cdot 7^2 + 7 \cdot 5^2 = 5 \cdot 7 \cdot (5+7) = 35 \cdot 12$. Since $C(0) = 84 \cdot 12$ and $C(12) = 60 \cdot 12$, $C(7)$ is the minimum, and the minimum cost is $35 \cdot 12$.

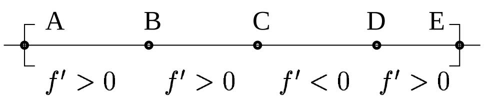
1^{st} derivative test

$C'(x) < 0$ if $x < 7$ and $C'(x) > 0$ if $x > 7$, so $x = 7$ is a relative minimum. Since this is the only critical point in the interval, this relative minimum is also an absolute minimum for the function in the interval.

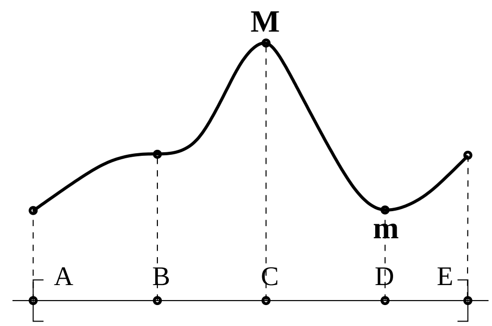
2^{nd} derivative test

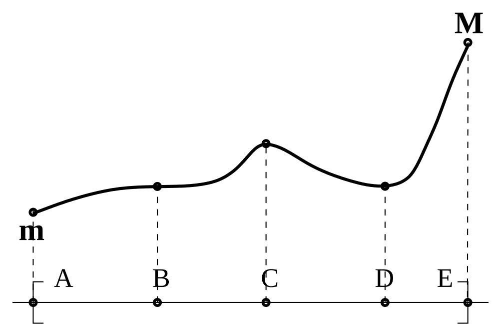
$C''(x) = 24$ which is positive, so the graph $y = C(x)$ is concave up, and any critical point must be a relative minimum. Since there is only one critical point, the relative minimum must be an absolute minimum.

The lecturer likes the "**0^{th} derivative test**" because it gives the minimum value without any additional computation. Also, if the situation were not as simple as this problem, the other tests wouldn't really help very much. For example, suppose there were three critical points in the interval. Suppose we also knew the signs of the first derivative away from the c.p.'s: lots of information. Something like this:

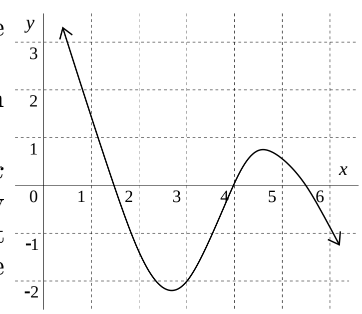


Then the function has a relative max at C and a relative min at D. It also has a horizontal tangent line at B, and this is an inflection point. **But** there is **not** enough information to find the absolute max and min of the function in the interval. Look at the two graphs shown to the right. The functions displayed both verify the first derivative information but the locations of the max and min are very different!





8. The graph of $y = f'(x)$, the *derivative* of the function $f(x)$, is shown to the right. Use the graph to answer the questions below.



The parts of this problem are *not* related but both parts use information from the graph of the derivative of $f'(x)$.

a) Use information from the graph of $f'(x)$ to find (as well as possible) the x where the *maximum value* of $f(x)$ in the interval $1 \le x \le 3$ will occur. Briefly explain using calculus why your answer is correct, including verification that the value of $f(x)$ you select is larger than $f(x)$ at *any* other number in the interval. **Answer** x is ≈ 1.6, the $x-$intercept of $y = f'(x)$ between 1 and 2. Call this x^*. $f'(x) < 0$ between x^* and 3, so the function decreases from x^* to 3: $f(x^*) > f(x)$ if $x^* < x \le 3$. Also, $f'(x) > 0$ for x between 1 and x^*, so $f(x)$ is increasing in $[1, x^*]$. Therefore $f(x^*) > f(x)$ if $1 \le x < x^*$.

b) Suppose that $f(3) = 5$. Use information from the graph and the tangent line approximation for $f(x)$ to find an approximate value of $f(3.04)$. Briefly explain using calculus and information from the graph why your approximate value for $f(3.04)$ is greater than or less than the exact value of $f(3.04)$. **Answer** Linear approximation gives $f(3.04) \approx f(3) + f'(3)(.04)$. The graph supplies $f'(3) = -2$ so $f(3.04) \approx 5 + (-2)(.04) = 4.92$. The tangent line to $y = f'(x)$ at $x = 3$ has slope > 0 (the graph of $y = f'(x)$ is *increasing* near $x = 3$) so the derivative of $f'(x)$ is positive there: $f''(x) > 0$ near $x = 3$, and $y = f(x)$ is concave up near $x = 3$. The approximate value is *less than* the exact value since the tangent line will lie below the graph of $y = f(x)$.