1. Choose an appropriate starting guess and then use three iterations of Newton's method to find the smallest positive solution to

$$\frac{1}{1+x^2} = \tan x$$

How many positive solutions to this equation are there? Why? What would you guess is true about the spacing and location of positive solutions to this equation as $x \to \infty$? (Pictures will help answer this question; explain your conclusions in sentences, referring to these pictures as needed.)

2. `This is probably how your calculator computes multiplicative inverses.`
In this problem, $a = 1.2345$ and $f(x) = \frac{1}{x} - a$.The object of this problem is to study the Newton's method iteration for finding $\frac{1}{a}$.*
a) Write the Newton's method iteration scheme for $f(x)$. That is, write a formula showing how an old guess G, for a root of $f(x) = 0$ is changed to a new and perhaps better guess, N. Please simplify the formula as much as possible so that it contains *no divisions.*

b) Suppose $x_0 = 1$ is the initial guess. How many iterations (repetitions) of Newton's method are needed to get $\frac{1}{a}$ to 10 digit accuracy? Note: $f(x) = 0$ when $x =$ ___.**

c) Now consider *any* starting point x_0 for Newton's method in this problem. Color x_0 **green** if the iteration of Newton's method converges to the only root of $f(x)$. Color x_0 **red** if the iteration of Newton's method does *not* converge to that root. Find an example of a red x_0 and a green x_0.

d) Continue your experimentation, supplemented with appropriate graphical and algebraic analysis. Find all red x_0's and all green x_0's. Discuss the solution as well as you can.***

3. The following statements are true facts:

$$(10{,}000{,}000{,}000)^{\left(\frac{1}{10{,}000{,}000{,}000}\right)} \approx 1.\ 00000\ 00\underline{023}\ \underline{02585}\ \underline{09}564$$

and

$$\ln 10 \approx \underline{2}.\underline{30258}\ \underline{509}29$$

Explain the amazing coincidence of the digits.

Hint Approximate e^x when x is small.

4. Suppose you **know** that $f'(x) = \dfrac{2}{1+x^4} - \dfrac{3}{4+x^4}$. Is $f(0) < f(1)$?

More information about this problem follows.

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* To end the suspense, $\frac{1}{a}$ is .81004455245038477116 to twenty place accuracy.

** You fill in the blank.

*** Have fun with numbers, but a picture will probably serve you better after a while. And some algebra following the picture will be even better.

Note #1 It is not likely at this time that you can write a formula for a f with this derivative (that can be done, and such f's have very complicated formulas). So you will have to make some *indirect* argument, just using the information you have about f'. Write out **two verifications** of your answer, one an algebraic argument using the formula for f' and the other, a geometric argument, using a graph of f' (which can be plotted on a calculator).

Note #2 Here is such a function:

$$f(x) = \frac{\sqrt{2}}{4}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}x+1) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}x-1)$$
$$+\frac{3}{16}\ln(x^2-2x+2) - \frac{3}{8}\arctan(x-1) - \frac{3}{16}\ln(x^2+2x+2) - \frac{3}{8}\arctan(x+1)$$

Does this formula, which should be checked if it is used, help, or is studying the derivative easier?

5. a) Suppose you know that $h'(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5$. What are the critical numbers of h? Which of them are local extrema, and what kind of local extrema are they?

b) Suppose you know that $k'(x) = x(x-1)^{2/3}(x-2)^{3/5}(x-3)^{4/7}$. What are the critical numbers of k? Which of them are local extrema, and what kind of local extrema are they?

Note You are *not* asked to compute h and k explicitly.