640:151:07–09 **Answers to version A of the First Exam** **10/14/2006**

Here are answers that would earn full credit. Other methods may also be valid.

(12) 1. Compute the derivatives of the functions shown. In this problem, you may write only the answers and get full credit. *Please* do not "simplify" the answers! a) $3+5\sqrt{x}-12x^7$ **Answer** $\frac{5}{2}x^{-1/2}-12(7x^6)$.

b) $e^{\left(\frac{2x+1}{3x^2-1}\right)}$ **Answer** $e^{\left(\frac{2x+1}{3x^2-1}\right)}\left(\frac{2(3x^2-1)-(6x)(2x+1)}{(3x^2-1)^2}\right)$.

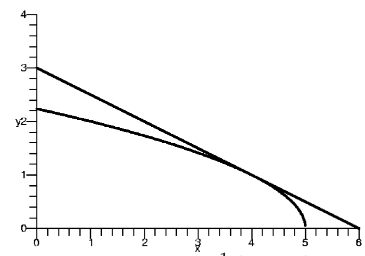
c) $\tan(5x)\left(\cos(x)\right)^3$ **Answer** $\left((\sec(5x))^2 5\right)\left(\cos(x)\right)^3+\tan(5x)\left(3(\cos(x))^2(-\sin(x))\right)$.

(6) 2. Compute the exact value of second derivative of $\sin(g(x))$ at $x=2$, assuming that $g(2)=\frac{\pi}{4}$, $g'(2)=5$, and $g''(2)=3$.

Answer If $f(x)=\sin(g(x))$, then $f'(x)=\cos(g(x))g'(x)$, using the Chain Rule. Now, using both the Chain Rule and the Product Rule, $f''(x)=-\sin(g(x))\left(g'(x)\right)^2+\cos(g(x))g''(x)$. When $x=2$, the result is $-\sin(g(2))\left(g'(2)\right)^2+\cos(g(2))g''(2)=-\sin(\frac{\pi}{3})\left(5\right)^2+\cos(\frac{\pi}{3})3=-(\frac{\sqrt{3}}{2})25+(\frac{1}{2})3$.

(14) 3. a) Use the definition of derivative combined with algebraic manipulation and standard properties of limits to compute the derivative of $f(x)=\sqrt{5-x}$. **Comment** Do **NOT** use l'Hôpital's rule.

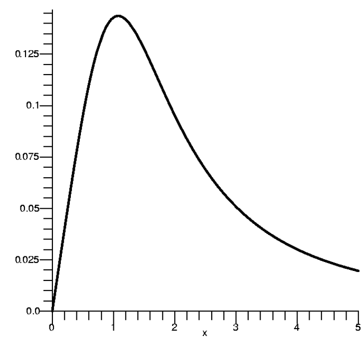
Answer $\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{5-(x+h)}-\sqrt{5-x}}{h}=\frac{\sqrt{5-(x+h)}-\sqrt{5-x}}{h}\cdot\frac{\sqrt{5-(x+h)}+\sqrt{5-x}}{\sqrt{5-(x+h)}+\sqrt{5-x}}$ $=\frac{(5-(x+h))-(5-x)}{h(\sqrt{5-(x+h)}+\sqrt{5-x})}=\frac{-h}{h(\sqrt{5-(x+h)}+\sqrt{5-x})}=\frac{-1}{\sqrt{5-(x+h)}+\sqrt{5-x}}$. We can now compute $\lim_{h\to 0}\frac{-1}{\sqrt{5-(x+h)}+\sqrt{5-x}}=-\frac{1}{2\sqrt{5-x}}$, and this is $f'(x)$.



b) Find an equation of the line tangent to $y=\sqrt{5-x}$ when $x=4$. You may and should use your answer to a). Sketch both this tangent line and $y=\sqrt{5-x}$ on the coordinate axes to the right.

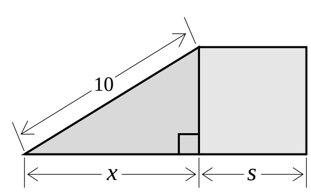
Answer $f(4)=\sqrt{5-4}=1$, and $f'(4)=-\frac{1}{2\sqrt{5-4}}=-\frac{1}{2}$. An equation for the line is $y-1=-\frac{1}{2}(x-4)$.

(12) 4. Suppose $f(x)=\frac{x}{5+2x^3}$. A portion of a computer-drawn graph of $y=f(x)$ is shown to the right. ~~The coordinate axes were lost in the printing process.~~ Find the exact coordinates of the top of the graph. (Do *not* simplify your answer!)



Answer If $f(x)=\frac{x}{5+2x^3}$, then $f'(x)=\frac{1(5+2x^3)-6x^2(x)}{(5+2x^3)^2}$. $f'(x)=0$ exactly when the top of the fraction is 0: $1(5+2x^3)-6x^2(x)=0$ and this is $5=4x^3$ or $\left(\frac{5}{4}\right)^{1/3}=x$. The exact coordinates of the point are $\left(\left(\frac{5}{4}\right)^{1/3},\frac{\left(\frac{5}{4}\right)^{1/3}}{5+2\left(\left(\frac{5}{4}\right)^{1/3}\right)^3}\right)$.

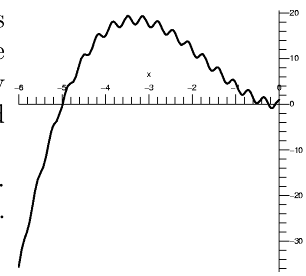
(12) 5. The length of the hypotenuse of a right triangle is 10 inches. One of the legs of the triangle has length x inches. The other leg of the triangle is one side of a square whose interior is outside the triangle. A diagram is shown to the right. Write a formula for $f(x)$, the <u>total area of the triangle and the square</u>, as a function of x. You may begin by finding a formula connecting the side length s of the square to x. What is the domain of this function when used to describe this problem? (The domain should be related to the problem statement.)



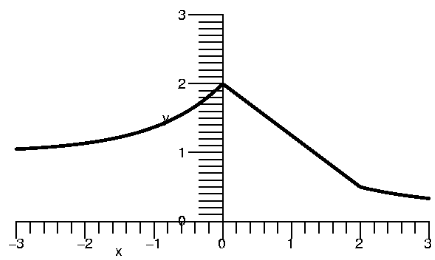
Answer The Pythagorean Theorem gives $s^2+x^2=10^2$, so that $s=\sqrt{10^2-x^2}$. The area of the square is $s^2=10^2-x^2$. The area of the triangle is $\frac{1}{2}$ BASE · HEIGHT $=\frac{1}{2}x\sqrt{10^2-x^2}$. Therefore a formula for $f(x)$ is $10^2-x^2+\frac{1}{2}x\sqrt{10^2-x^2}$. The domain for this problem is $[0,10]$.

(8) 6. Suppose $f(x)=x^3+5x^2+\cos(18x)$. Find a closed interval in which the equation $f(x)=0$ must have a solution. Give careful evidence supporting your assertion, including *specific results* from Math 151.

Answer $f(0) = 1$ and $f(-10) = -1{,}000 + 500 + \cos(18 \cdot (-10))$. Since the values of cosine are between -1 and 1, $f(-10) < 0$. The function f is continuous so the Intermediate Value Theorem implies that there is x in $[-10, 0]$ with $f(x) =$ any number between $f(-10)$ and $f(0)$. 0 is one such number because $f(-10) < 0$ and $f(0) > 0$. Therefore $f(x) = 0$ has a solution in the interval $[-10, 0]$.
Comment A computer-drawn graph of f on the interval $[-6, 0]$ is shown here. $f(-6) = -6(-6)^2+5(-6)^2+\cos(18\cdot(-6))$ which is negative using similar reasoning. I didn't expect three roots in the interval $[-6, 0]$.

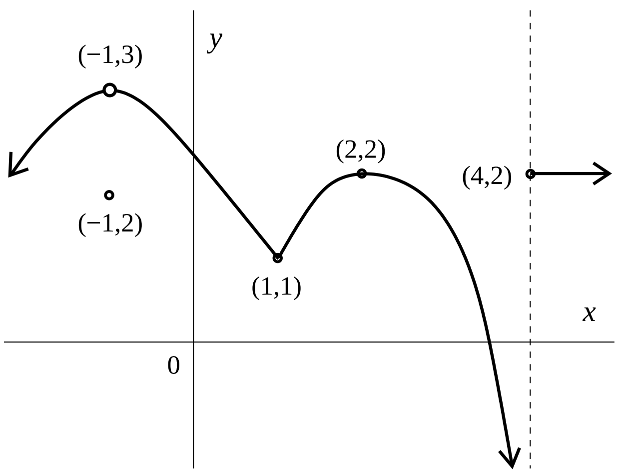


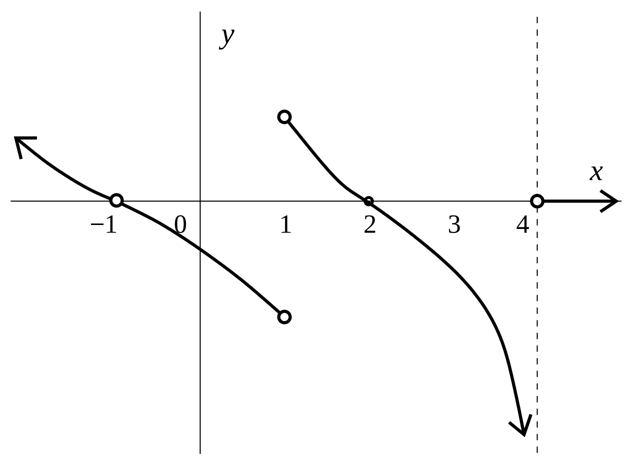
(12) 7. Suppose $f(x)=\begin{cases} 1+e^x & \text{for } x<0. \\ Ax+B & \text{for } 0\le x\le 2. \\ \frac{1}{x} & \text{for } x>2. \end{cases}$ a) Find values of A and B so that f will be continuous for all x.
Answer f is continuous for $x < 0$ and $0 < x < 2$ and $x > 2$ since in those intervals f is determined by formulas which are continuous for all numbers. We check behavior at 0 and at 2.
At $x = 0$ $f(0) = A(0)+B = B$; $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1+e^x = 1+e^0 = 2$; $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} Ax + B = B$. These will all agree if $B = 2$.
At $x = 2$ $f(2) = A(2) + B = 2A + B$; $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} Ax + B = 2A+B$; $\lim_{x\to 2^+} f(x) = \lim_{x\to 0^-} \frac{1}{x} = \frac{1}{2}$. These will all agree if $2A+B = \frac{1}{2}$.
Since $B = 2$, we know $2A = \frac{1}{2} - 2 = -\frac{3}{2}$ and so $A = -\frac{3}{4}$.
b) Sketch $y = f(x)$ on the axes below using the values of A and B found in a). [RIGHT]



(14) 8. Here is a graph of $y = f(x)$, a function whose domain is all real numbers. [BELOW]

a) Use these axes to sketch a graph of $y = f'(x)$ as accurately as possible. [BELOW]





b) For which x's can you conclude that f is *not* continuous? **Answer** $x = -1$ and $x = 4$.
c) For which x's can you conclude that f is *not* differentiable? **Answer** $x = -1$, $x = 1$, and $x = 4$.

(10) 9. Compute these limits. Supporting work for each answer *must* be given to earn full credit.
Comment For the "experts": do **NOT** use l'Hôpital's rule. Your results should be a specific number, $+\infty$, $-\infty$, or Does Not Exist.

a) $\lim_{x\to 0} \frac{x}{\frac{1}{x+3}+\frac{1}{x-3}}$ **Answer** $\frac{x}{\frac{1}{x+3}+\frac{1}{x-3}} = \frac{x}{\frac{(x-3)+(x+3)}{(x+3)(x-3)}} = \frac{x}{\frac{2x}{(x+3)(x-3)}} = \frac{x(x+3)(x-3)}{2x} = \frac{(x+3)(x-3)}{2}$, and, finally, $\lim_{x\to 0} \frac{(x+3)(x-3)}{2} = -\frac{9}{2}$.

b) $\lim_{x\to 2^-} \frac{|x-2|}{x^2-4}$ **Answer** If $x < 2$, then $x - 2 < 0$ and $|x - 2| = -(x - 2) = 2 - x$. Then $\frac{|x-2|}{x^2-4} = \frac{2-x}{(x-2)(x+2)} = -\frac{1}{x+2}$, and $\lim_{x\to 2^-} -\frac{1}{x+2} = -\frac{1}{4}$.

c) $\lim_{x\to 4} \frac{x-1}{x^2-9}$. **Answer** The function $f(x) = \frac{x-1}{x^2-9}$ is continuous for $x \neq \pm 3$. Therefore $\lim_{x\to 4} f(x) = f(4) = \frac{4-1}{4^2-9} = -\frac{3}{7}$.