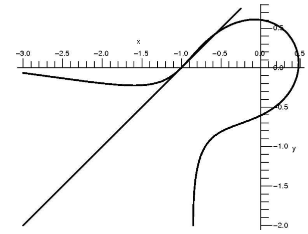
Here are answers that would earn full credit. Other methods may also be valid.

(12) 1. The program `Maple` displays the image shown to the right when graphing the equation $\ln(x^2+y^2) - xy + x = -1$.

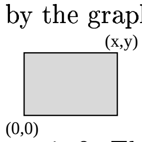


a) Verify by substitution in the equation that the point $(-1,0)$ is on the graph of the equation. **Answer** Plug in $x=-1$ and $y=0$. The result is $\ln((-1)^2+0^2) - (-1)0 + (-1) = -1$ which is $\ln(1) - 1 = -1$, certainly true since $\ln(1)=0$.

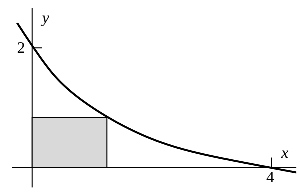
b) Use calculus to find an equation for the line tangent to the graph at the point $(-1,0)$. **Answer** $\frac{d}{dx}$ the equation: $\left(\frac{1}{x^2+y^2}\right)(2x+2yy') - (y+xy') + 1 = 0$. If $x=-1$ and $y=0$ this becomes $\left(\frac{1}{(-1)^2+0^2}\right)(2\cdot(-1)+2(0)y') - (0+(-1)y') + 1 = 0$ so that $y'=1$. This is the slope of the tangent line. The line goes through $(-1,0)$ so an equation of the tangent line is $y = 1(x+1)$

c) Sketch this tangent line in the appropriate place on the image displayed. **Answer** See the graph shown.

(12) 2. Find the area of the largest rectangle that can be inscribed in the region bounded by the graph of $y = \frac{4-x}{2+x}$ and the coordinate axes. **Answer** A, the box's area, is

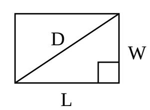


$xy = x\cdot\left(\frac{4-x}{2+x}\right) = \frac{4x-x^2}{2+x}$. Also $0 \le x \le 4$, $A(0)=0$, and $A(4)=0$. The derivative is $A'(x) = \frac{(4-2x)(2+x)-(4x-x^2)}{(2+x)^2}$. We need to know when the top is 0. The top is $8-4x-x^2$ with roots (by quadratic formula) $-2\pm\sqrt{12}$. The The root $-2-\sqrt{12}$ is not in $[0,4]$: discard it. $A(-2+\sqrt{12})$ is positive, so (compare with the endpoint values) this is the maximum. The largest area is $\frac{(-2+\sqrt{12})(4+2-\sqrt{12})}{\sqrt{12}}$.



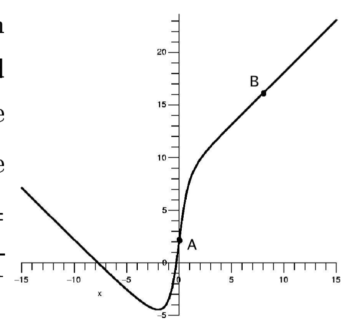
(12) 3. At a certain time, a rectangle has **L**ength equal to 5 inches and **W**idth equal to 7 inches. Also at that time, the **L**ength is <u>increasing</u> at .3 inches per second and the **W**idth is <u>decreasing</u> at .4 inches per second.

a) Compute the area of the rectangle at the certain time. Is the area increasing or decreasing at that time, and at what rate? **Answer** Suppose A is the area. Then $A = LW$ so that $A' = L'W + WL'$. At the "certain time", $A = 5\cdot 7 = 35$ and $A' = .3(7) + 5(-.4) = .1$. The area is increasing.



b) Compute the length of the diagonal of the rectangle at the certain time. Is this length increasing or decreasing at that time, and at what rate? **Answer** Suppose D is the length of the rectangle's diagonal. Then $D = \sqrt{L^2+W^2}$ and $D' = \frac{1}{2\sqrt{L^2+W^2}}(2LL' + 2WW')$. At the "certain time", $D = \sqrt{5^2+7^2} = \sqrt{74}$ and $D' = \frac{1}{\sqrt{74}}(5(.3)+7(-.4)) = -\frac{1.3}{\sqrt{74}}$. The length is decreasing.

(20) 4. In this problem, $f(x) = \frac{x^2+8x+2}{\sqrt{x^2+1}}$. A `Maple` graph of $y = f(x)$ is shown to the right. Also, the following equations are true: $f'(x) = \frac{x^3+8}{(x^2+1)^{3/2}}$ and $f''(x) = \frac{3x(x-8)}{(x^2+1)^{5/2}}$. a) Compute $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Give evidence (not just the graph!) supporting your assertions. **Answer** $\lim_{x\to\infty} \frac{x^2+8x+2}{\sqrt{x^2+1}}$ is the indeterminate form $\frac{\infty}{\infty}$ so L'H gives $\lim_{x\to\infty} \frac{2x+8}{\frac{2x}{2\sqrt{x^2+1}}} = \lim_{x\to\infty} \frac{(2x+8)\sqrt{x^2+1}}{x} = \lim_{x\to\infty}(2+\frac{8}{x})\sqrt{x^2+1} = +\infty$. The $-\infty$ limit also results in $+\infty$ since $\sqrt{x^2+1}$ is always positive. Other methods also can be used.



b) Where is f increasing? Where is f decreasing? Support your assertions using results from the course and information supplied. **Answer** The bottom of $f'(x) = \frac{x^3+8}{(x^2+1)^{3/2}}$ is always positive. Then f is increasing where $x^3 > -8$ or $x > -2$. Similarly, f is decreasing where $x^3 < -8$ or $x < -2$.

c) Exactly how many roots does the equation $f(x) = 100$ have? Do *not* try to compute these roots but use results from the course and your answers to a) and b) to verify your answer. **Answer** The equation has exactly two roots. The limits shown in a) imply that eventually both the left and right sides of the graph are above 100, so continuity (Intermediate Value Theorem) implies there's a root on each "side" (positive and negative x). But f is increasing in $(-2,\infty)$ so the equation can have only one root in $(-2,\infty)$. Similarly, f' decreasing in $(-\infty,-2)$ so the equation can have only one root in $(-\infty,-2)$.

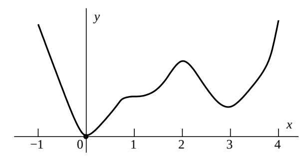
d) Where is the graph of $y = f(x)$ concave up and concave down? Label any points of inflection <u>on the graph</u>. (You need not give support for your answer here but indicate intervals clearly below, and label the graph as needed.) **Answer** Look at the sign of f''. The graph is concave up in $(0,8)$. It is concave down in $(-\infty,0)$ and in $(8,\infty)$. The points of inflection are noted (A and B) on the graph.

(16) 5. Suppose $f'(x) = x(x-1)^2(x-2)^3(x-3)$.

a) Where are the critical points of f and what are their types? Enter the information in the table below. The types should be selected from the words MAXIMUM, MINIMUM, and NEITHER. *Briefly* give reasoning which supports your assertions about this function in the space after the table.

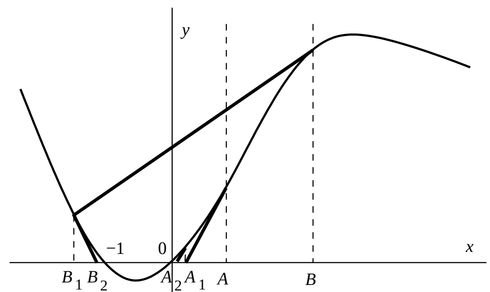
Critical point: $x =$	0	1	2	3
Type of critical point	MINIMUM	NEITHER	MAXIMUM	MINIMUM

Explanation f is increasing if f' is positive and decreasing if f' is negative. The exponents' parity (even/odd) shows that f' changes sign at 0, 2, and 3, but not at 1. Therefore we only need to check a "starting" sign for f', such as $f'(-1) = (-1)(-2)^2(-3)^3(-4) < 0$ which establishes the {in|de} nature of each c.p. (First Derivative Test). b) Suppose you know that $f(0) = 0$. Sketch a qualitatively correct graph of $y = f(x)$ on the interval $[-1, 4]$ using the axes below. To help, a point on the graph at $(0, 0)$ has already been drawn.



(10) 6. A graph of $y = f(x)$ is shown below. The equation $f(x) = 0$ has solutions when $x = 0$ and when $x = -1$.

a) Suppose that Newton's method is applied to find a solution of $f(x) = 0$ with starting value $x = A$. Show on the graph's x–axis as well as you can (with labels!) the result of the first two iterations A_1 and A_2 of Newton's method. What is the limit of the sequence of iterations $\{A_n\}$ that begins with $x = A$? Draw appropriate supporting evidence for your A_1 and A_2 on the graph. Your limit assertion needs no supporting evidence. **Answer** $\lim_{n\to\infty} A_n = 0$



b) Suppose that Newton's method is applied to find a solution of $f(x) = 0$ with starting value $x = B$. Show on the graph's x–axis as well as you can (with labels!) the result of the first two iterations B_1 and B_2 of Newton's method. What is the limit of the sequence of iterations $\{B_n\}$ that begins with $x = B$? Draw appropriate supporting evidence for your B_1 and B_2 on the graph. Your limit assertion needs no supporting evidence. **Answer** $\lim_{n\to\infty} B_n = -1$

(10) 7. Compute these limits. Supporting work for each answer *must* be given to earn full credit. **Comment** Your results should be a specific number, $+\infty$, $-\infty$, or Does Not Exist.

a) $\lim_{x\to 0} \frac{e^{\left(5x^2\right)}-1}{x^2-x^3}$. **Answer** Plug in $x = 0$ and get $\frac{e^0-1}{0-0} = \frac{0}{0}$: eligible for L'H. So consider $\lim_{x\to 0} \frac{e^{\left(5x^2\right)}(10x)}{2x-3x^2}$. Plug in $x = 0$ and get $\frac{e^0(10\cdot 0)}{0-0} = \frac{0}{0}$: eligible for L'H. So consider $\lim_{x\to 0} \frac{e^{\left(5x^2\right)}(10x)^2+e^{\left(5x^2\right)}10}{2-6x}$. When $x = 0$ we get $\frac{e^0(0)+e^0(10)}{2-6\cdot 0} = \frac{10}{2} = 5$.

b) $\lim_{x\to 0} (1 + k \sin x)^{1/x}$ (here k is a positive constant) **Answer** Take ln and rearrange. Consider $\lim_{x\to 0} \frac{\ln(1+k\sin x)}{x}$. Plug in $x = 0$ and get $\frac{\ln(1+k\sin(0))}{0} = \frac{0}{0}$: eligible for L'H. So consider $\lim_{x\to 0} \frac{\frac{1}{1+k\sin x}(k\cos x)}{1}$. When $x = 0$ we get k. Exponentiate to get the original limit's value, which is e^k.

c) $\lim_{x\to 5} \frac{x^2-25}{x+5}$. Plug in $x = 5$ and get $\frac{0}{10}$, so the limit is 0 "by continuity".

(8) 8. A car moves along a road from the town of FROG to the town of TOAD. Its speed is always between 40 and 60 mph. The car leaves FROG at 9 AM and enters TOAD at 10:30 AM.

Write a mathematical model of this situation. Your model should include a function with a domain you define, with properties of the function stated carefully. Use this model together with *specific results* from Math 151 to obtain some estimate of the distance along the road from FROG to TOAD.

Answer Suppose $s(t)$ is the distance in miles on the road that the car has traveled from FROG. Here t is time, measured in hours, from 9 AM. We know that $s(0) = 0$ and $s(1.5) = U$ (U is unknown). We also suppose that $s(t)$ is differentiable, $s'(t) = v(t)$, and we know $40 \le v(t) \le 60$ for all t. The Mean Value Theorem applies: there is t_0 inside the time interval so that $s'(t_0) = \frac{s(1.5)-s(0)}{1.5-0} = \frac{U}{1.5}$. Then we know $40 \le \frac{U}{1.5} \le 60$ and therefore $60 \le U \le 90$. TOAD is between 60 and 90 miles from FROG along the road.