Review Final Examination
Mathematics 151

Fall Semester 2007

1. Calculate the following limits:

(a) $\lim_{x\to e} \frac{e^x - x^e}{(x-e)^2}$ (b) $\lim_{x\to\infty} (1+x)^{1/x}$ (c) $\lim_{x\to\infty} (\sqrt{x^2+x} - \sqrt{x^2-x})$

2. Find the following derivatives:

(a) $\frac{d}{dx}\left(\frac{x^2+x+1}{x^3+x}\right)$ (b) $\frac{d}{dx}(x\tan(e^{-x}))$ (c) $\frac{d}{dx}((1+x+x^2)^x)$ (d) $\frac{d}{dx}\left(\int_0^x \sqrt{1+t^4}\,dt\right)$

3. A function is defined by: $f(x) = \begin{cases} x^2 + Ax + B, & \text{if } |x| \le 1, \\ 1/x, & \text{if } 1 < |x|, \end{cases}$ with A, B constants.

(a) Find A, B so that f is continuous everywhere.
(b) For those values of A, B, sketch the graph of $y = f(x)$ and determine those points at which f fails to be differentiable.

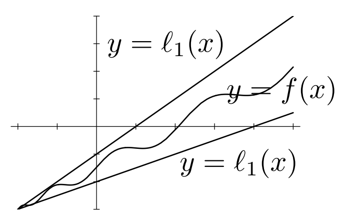
4. Use the definition of the derivative directly to calculate $f'(x)$, where $f(x) = \frac{1}{(2x+1)^2}$.

5. A function $y = f(x)$ is defined implicitly by the equation: $x^3 + 3x^2y + y^3 = 5$.

(a) Find $\frac{dy}{dx}$ in terms of x and y.
(b) Find the equation of the tangent line to the above curve at the point $(1, 1)$.
(c) At which points (x, y) of the curve is the tangent line horizontal?

6. A balloon is being inflated so that, at a particular time, its volume is increasing at a rate of 5 cubic inches per second and its surface area is increasing at a rate of 3 square inches per second. What is the radius of the balloon at that particular time?

7. Suppose that $y = \ell_1(x)$ and $y = \ell_2(x)$ are lines, both of which contain the point $(-2, -3)$. Suppose that the first line has slope $1/2$ and the second has slope 1.
(a) Find the equations $y = \ell_1(x)$, $y = \ell_2(x)$.
(b) Suppose that f is a differentiable function with $f(-2) = -3$ and, if $-2 \le x$, $1/2 \le f'(x) \le 1$. Use the Mean Value Theorem to show that if $-2 \le x$, then $\ell_1(x) \le f(x) \le \ell_2(x)$, as shown in the picture.



8. Assume that f is a differentiable function having six critical points: $-5, -3, 0, 1, 3, 6$. Suppose that the sign of $f'(x)$ between successive critical points is given by the following table:

Interval	$(-\infty, -5)$	$(-5, -3)$	$(-3, 0)$	$(0, 1)$	$(1, 3)$	$(3, 6)$	$(6, \infty)$
Sign of $f'(x)$	+	−	−	+	+	−	+

Determine which of the critical points are local maxima, which are local minima, and which are neither.

9. Sketch the graph of the following functions, including all vertical and horizontal asymptotes, critical points and inflection points. Also, find the domain and range. Give exact answers rather than numerical approximations.

(a) $y = f(x) = \frac{1}{x} + \ln x$

(b) $y = g(x) = x^3 e^{-x^2/2}$

10. Suppose that f is a differentiable function and that $f(1) = 3$, $f'(1) = -2$.
(a) Find the linear or tangent line approximation, $L(x)$, of $f(x)$ at $a = 1$.

(b) Use the linear approximation of (a) to estimate $f(1.05)$.
(c) Let $g(x) = x^2 f(x)$. Find the linear approximation of $g(x)$ at $a = 1$.

11. Determine the following integrals:

(a) $\displaystyle\int \frac{x^2+x+1}{\sqrt{x}}\,dx$ (b) $\displaystyle\int_0^1 \sqrt{x+1}\,dx$ (c) $\displaystyle\int x^2 \sin(x^3)\,dx$ (d) $\displaystyle\int_1^e \frac{\cos(\ln x)}{x}\,dx$

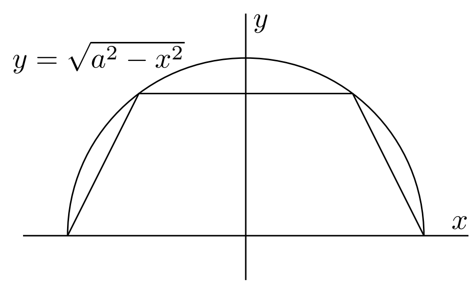
12. Suppose that $x(t)$ is the position of a particle moving on the x-axis. The acceleration $a(t)$ of the particle is given by: $a(t) = \dfrac{d^2x}{dt^2} = -24t^2 + 96t.$

Assume that $x(0) = 0$ and $v(0) = x'(0) = 0$.

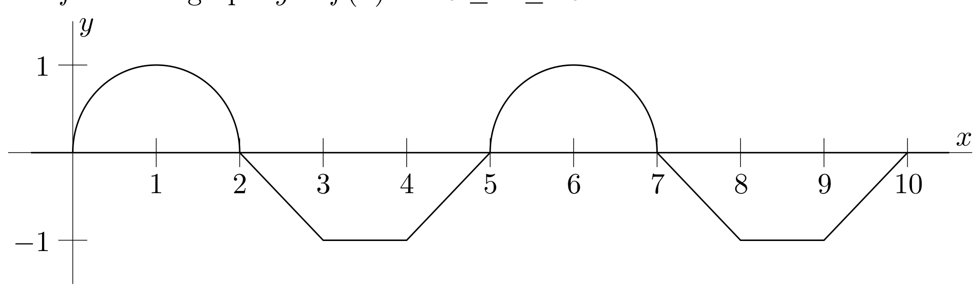
(a) Find $x(t)$ as a function of t.
(b) The particle moves down the positive x-axis and then returns to the origin. How far down the x-axis does the particle go?

13. Suppose that a is a positive constant. A trapezoid is inscribed in the semi-circle $y = \sqrt{a^2 - x^2}$ with the base of the trapezoid on the x-axis and the upper corners on the semi-circle as shown in the picture below.

Find the maximum possible area of the trapezoid.



14. A function f has the graph $y = f(x)$ for $0 \le x \le 10$ as shown below.



The arcs above the intervals $[0, 2]$ and $[5, 7]$ are semi-circles.

Find the integrals:

(i) $\displaystyle\int_0^2 f(x)\,dx$ *(ii)* $\displaystyle\int_0^5 f(x)\,dx$ *(iii)* $\displaystyle\int_0^7 f(x)\,dx$ *(iv)* $\displaystyle\int_0^{10} f(x)\,dx$ *(v)* $\displaystyle\int_3^9 f(x)\,dx$

15. With $f(x) = \dfrac{1}{x}$, write out the Riemann sums $\displaystyle\sum_{i=1}^{4} f(x_i^*)\Delta x_i$ for the integral $\displaystyle\int_1^3 f(x)\,dx$, where the interval $[1, 3]$ is subdivided into four subintervals of equal length and
(a) x_i^* is chosen as the left endpoint of each subinterval;
(b) x_i^* is chosen as the right endpoint of each subinterval;
(c) x_i^* is chosen as the midpoint of each subinterval.

16. A region R in the first quadrant is bounded by the line $y = x$ and the parabola $y = -x^2+5x-3$. Find the area of R.