Some answers to the integration quiz for 152:5-7 and 9–11 12/1/2007

1. Compute $\int xe^{-3x}\,dx$.

Answer Use integration by parts, with $\begin{cases} u=x \\ dv=e^{-3x}\,dx \end{cases}$ so $\begin{cases} du=dx \\ dv=-\frac{1}{3}e^{-3x}\,dx \end{cases}$. Then $\int u\,dv = uv-\int v\,du$ is $\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x}+\frac{1}{3}\int e^{-3x}\,dx = \underline{-\frac{1}{3}xe^{-3x}-\frac{1}{9}e^{-3x}+C}$.

2. Compute $\int x^2\sqrt{x+1}\,dx$.

Answer *First method*

Use integration by parts, with $\begin{cases} u=x^2 \\ dv=\sqrt{x+1}\,dx \end{cases}$ so $\begin{cases} du=2x\,dx \\ v=\frac{2}{3}(x+1)^{3/2} \end{cases}$ and $\int u\,dv = uv-\int v\,du$ gives $\int x^2\sqrt{x+1}\,dx = \frac{2}{3}x^2(x+1)^{3/2}-\frac{4}{3}\int x(x+1)^{3/2}\,dx$. Now use integration by parts on $\int x(x+1)^{3/2}\,dx$ with $\begin{cases} u=x \\ dv=(x+1)^{3/2}\,dx \end{cases}$ so $\begin{cases} du=dx \\ v=\dfrac{2}{5}(x+1)^{5/2} \end{cases}$ and $\int u\,dv = uv-\int v\,du$ turns into $\int x(x+1)^{3/2}\,dx = \frac{2}{5}x(x+1)^{5/2}-\frac{2}{5}\int(x+1)^{5/2}\,dx = \frac{2}{5}x(x+1)^{5/2}-\frac{4}{35}(x+1)^{7/2}$. Combine the results to get the whole answer: $\int x^2\sqrt{x+1}\,dx = \underline{\frac{2}{3}x^2(x+1)^{3/2}-\frac{4}{3}\left(\frac{2}{5}x(x+1)^{5/2}-\frac{4}{35}(x+1)^{7/2}\right)+C}$.

Second method

Use substitution on $\int x^2\sqrt{x+1}\,dx$ with $u=x+1$. Then $du=dx$ and $u-1=x$ so $(u-1)^2=x^2$. The integral becomes $\int(u-1)^2u^{1/2}\,du = \int(u^2-2u+1)u^{1/2}\,du = \int u^{5/2}-2u^{3/2}+u^{1/2}\,du = \frac{2}{7}u^{7/2}-\frac{4}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C = \underline{\frac{2}{7}(x+1)^{7/2}-\frac{4}{5}(x+1)^{5/2}+\frac{2}{3}(x+1)^{3/2}+C}$.

Third method

Use substitution on $\int x^2\sqrt{x+1}\,dx$ with $u^2=x+1$ because then $u=\sqrt{x+1}$. Then $x=u^2-1$ and $dx=2u\,du$ so the integral becomes $\int(u^2-1)^2u(2u\,du) = \int(u^4-2u^2+1)2u^2\,du = \int 2u^6-4u^4+2u^2\,du = \frac{2}{7}u^7-\frac{4}{5}u^5+\frac{2}{3}u^3+C =$
$\underline{\frac{2}{7}(\sqrt{x+1})^7-\frac{4}{5}(\sqrt{x+1})^5+\frac{2}{3}(\sqrt{x+1})^3+C}$.

Comment `Maple`'s answer is $\underline{\frac{2}{105}(x+1)^{3/2}(8-12x+15x^2)}$. All of these answers must be the same!

3. Compute $\int(\sin x)^3(\cos x)^2\,dx$.

Answer Observe: $(\sin x)^3(\cos x)^2 = (\sin x)^2(\cos x)^2\sin x = \left(1-(\cos x)^2\right)(\cos x)^2\sin x$. Then I "guess" the substitution $u=\cos x$. Then $du=-\sin x\,dx$ so $\int(\sin x)^3(\cos x)^2\,dx = \int\left(1-(\cos x)^2\right)(\cos x)^2\sin x\,dx = -\int(1-u^2)u^2\,du = \int u^2-u^4\,du = -\frac{1}{3}u^3+\frac{1}{5}u^5+C = \underline{-\frac{1}{3}(\cos x)^3+\frac{1}{5}(\cos x)^5+C}$.

Comment `Maple`'s answer is different. I suspect it used a reduction formula for the powers of sine or cosine (that is, it used integration by parts first).