640:152: $\left\{ \begin{matrix} 5, & 6, & 7 \\ 9, & 10, & 11 \end{matrix} \right\}$ **Answers to version A of the First Exam** **2/21/2007**

Here are answers that would earn full credit. Other methods may also be valid.

(12) 1. Suppose $\mathcal{R}$ is the region bounded by $y = e^x$, $x = 0$, $x = 2$, and $y = 0$.

a) Find the volume of the solid that results from rotating $\mathcal{R}$ around the x-axis.

Answer $\pi \int_0^2 (e^x)^2 dx = \pi \int_0^2 e^{2x} dx = \frac{\pi}{2} e^{2x} \Big]_0^2 = \frac{\pi}{2} e^4 - \frac{\pi}{2}$.

b) Find the volume of the solid that results from rotating $\mathcal{R}$ around the y-axis.

Answer The volume is $2\pi \int_0^2 x e^x dx$. An antiderivative of xe^x can be obtained using integration by parts. If $u = x$ and $dv = e^x dx$, then $du = dx$ and $v = e^x$ so $uv - v\,du$ is $xe^x - \int e^x dx = xe^x - e^x + C$. Therefore the volume is $2\pi\,(xe^x - e^x)]_0^2 = 2\pi\left(2e^2 - e^2\right) - 2\pi\,(-1)) = 2\pi(e^2 + 1)$.

(12) 2. Compute $\int_1^\infty \frac{\ln x}{x^3}\,dx$.

Answer Use integration by parts to get an antiderivative of $\frac{\ln x}{x^3}$. Here $u = \ln x$ and $dv = \frac{1}{x^3}\,dx$ so $du = \frac{1}{x}\,dx$ and $v = -\frac{1}{2x^2}$. Then $uv - v\,du$ is $-\frac{\ln x}{2x^2} - \int -\frac{1}{2x^3} dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$ (three $-$'s are "built into" the last $-$!). Then (for A positive) $\int_1^A \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2}\Big]_1^A = -\frac{\ln A}{2A^2} - \frac{1}{4A^2} - \left(-\frac{\ln 1}{2\cdot 1^2} - \frac{1}{4\cdot 1^2}\right)$. Now as $A \to \infty$, certainly $\frac{1}{4A^2} \to 0$. The limit of $\frac{\ln A}{2A^2}$ needs L'H since both $\ln A$ and A^2 go to ∞. But $\lim_{A\to\infty} \frac{\ln A}{2A^2} \overset{\text{L'H}}{=} \lim_{A\to\infty} \frac{\frac{1}{A}}{4A} = \lim_{A\to\infty} \frac{1}{4A^2} = 0$. So the limit of $\int_1^A \frac{\ln x}{x^3}\,dx$ as $A \to \infty$ is $-\left(-\frac{\ln 1}{2\cdot 1^2} - \frac{1}{4\cdot 1^2}\right)$ which is $\frac{1}{4}$.

(12) 3. Verify that $\int_1^2 \frac{5x^2+11x+4}{x(x+1)(x+2)}\,dx = \ln(12)$.

Answer Use partial fractions. The bottom is factored, and therefore we write $\frac{5x^2+11x+4}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} = \frac{A(x+1)(x+2)+Bx(x+2)+Cx(x+1)}{x(x+1)(x+2)}$ for some constants A, B, and C. Then $5x^2 + 11x + 4 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$. If $x = 0$, $4 = 2A$ so $\underline{A = 2}$. If $x = -1$, $5(-1)^2 + 11(-1) + 4 = B(-1)(1)$, so $\underline{B = 2}$. If $x = -2$, $5(-2)^2 + 11(-2) + 4 = C(-2)(-1)$ so $\underline{C = 1}$. Therefore $\int \frac{5x^2+11x+4}{x(x+1)(x+2)}\,dx = \int \frac{2}{x} + \frac{2}{x+1} + \frac{1}{x+2}\,dx = 2\ln(x) + 2\ln(x+1) + \ln(x+2) + C$. The definite integral is $2\ln(x) + 2\ln(x+1) + \ln(x+2)]_1^2 = (2\ln(2) + 2\ln(3) + \ln(4)) - (2\ln(1) + 2\ln(2) + \ln(3)) = 2\ln(2) + \ln(3) = \ln(12)$.

(12) 4. Verify that $\int_0^1 x\arctan(x)\,dx = \frac{1}{4}\pi - \frac{1}{2}$.

Answer Use integration by parts. Here $u = \arctan(x)$ and $dv = x\,dx$ so that $du = \frac{1}{1+x^2}\,dx$ and $v = \frac{1}{2}x^2$. Therefore $\int x\arctan(x)\,dx = \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$. But $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$ so $\int \frac{x^2}{1+x^2}\,dx = x - \arctan x + C$ and $\int x\arctan(x)\,dx = \frac{1}{2}x^2\arctan(x) - \frac{1}{2}(x - \arctan x) + C$. And the definite integral: $\frac{1}{2}x^2\arctan(x) - \frac{1}{2}(x - \arctan x)\Big]_0^1 = \left(\frac{1}{2}1^2\arctan(1) - \frac{1}{2}(1 - \arctan 1)\right) - \left(\frac{1}{2}0^2\arctan(0) - \frac{1}{2}(0 - \arctan 0)\right) = \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$.

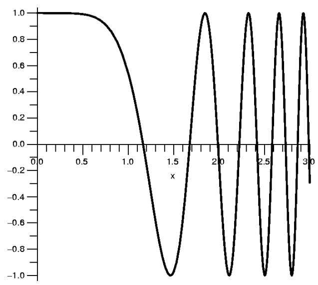
(12) 5. Compute $\int_0^1 \left(\sqrt{x} - 1\right)^6\,dx$.

Answer If $u = \sqrt{x} - 1$ then $u + 1 = \sqrt{x}$ and $(u+1)^2 = x$. So $2(u+1)\,du = dx$. Then $\int \left(\sqrt{x} - 1\right)^6\,dx = \int (u^6) 2(u+1)\,du = 2\int \left(u^7 + u^6\right)\,du = 2\left(\frac{u^8}{8} + \frac{u^7}{7}\right) + C = 2\left(\frac{\left(\sqrt{x}-1\right)^8}{8} + \frac{\left(\sqrt{x}-1\right)^7}{7}\right) + C$. And the definite integral: $2\left(\frac{\left(\sqrt{x}-1\right)^8}{8} + \frac{\left(\sqrt{x}-1\right)^7}{7}\right)\Bigg]_0^1 = 2\,(0) - \left(\frac{(-1)^8}{8} + \frac{(-1)^7}{7}\right) = \frac{2}{56} = \frac{1}{28}$.

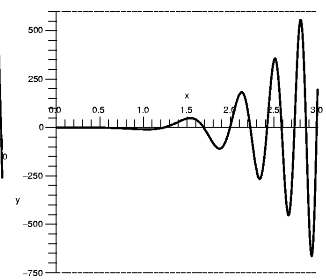
(14) 6. a) Write the Simpson's Rule estimate for $\int_0^3 \cos(x^3)\,dx$ with $n = 6$ subintervals.

Answer $\frac{\frac{1}{2}}{3}\left(1\cos(0^3) + 4\cos((.5)^3) + 2\cos(1^3) + 4\cos((1.5)^3) + 2\cos(2^3) + 4\cos((2.5)^3) + 1\cos(3^3)\right)$.

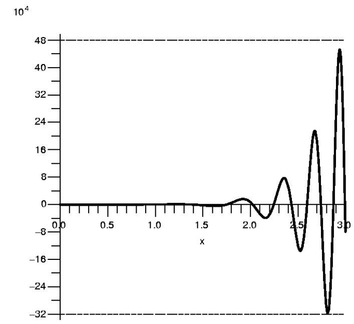
b) Below are graphs of $y = \cos(x^3)$ and of the second and fourth derivatives of this function on the interval $[0, 3]$. Assume that these graphs are correct. You may use information from these graphs to answer the following question. How many subdivisions are needed to estimate $\int_0^3 \cos(x^3)\,dx$ with the <u>Trapezoidal Rule</u> to an accuracy of 10^{-10}?



Graph of $y = \cos(x^3)$



Graph of the second derivative of $\cos(x^3)$

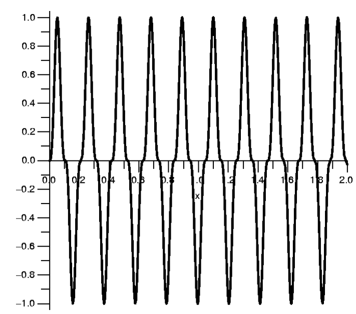


Graph of the fourth derivative of $\cos(x^3)$

Answer Use the middle graph above to get an overestimate of the absolute value of the second derivative of $\cos(x^3)$ on $[0,3]$: 750. Then the formula sheet states that the error for the Trapezoidal Rule will be less than $\frac{K(b-a)^3}{12n^2}$. Take $K = 750$ and $b - a = 3 - 0 = 3$. Then the error estimate becomes $\frac{750(3^3)}{12n^2}$. This will be less than 10^{-10} if $\frac{750(3^3)}{12n^2} < 10^{-10}$ (notice the direction of the inequality!) so n should be an integer greater than $\sqrt{\frac{750(3^3)10^{10}}{12}}$. (This is $\approx$4,100,000. If I had asked for Simpson's Rule with the same error, even though a *huge* fourth derivative bound of 480,000 would be used, n would be $\approx$9,000, a much smaller number!)

(12) 7. a) Suppose A is a positive real number and let m_A be the average value of $(\sin(Ax))^3$ on the interval $[0,2]$. Compute m_A.

Answer We need $\int (\sin(Ax))^3\,dx = \int (\sin(Ax))^2 \sin(Ax)\,dx$. If $u = \cos(Ax)$, then $(\sin(Ax))^2 = 1 - (\cos(Ax))^2 = 1 - u^2$ and $du = -A\sin(Ax)\,dx$. Therefore $\int (\sin(Ax))^3\,dx = -\frac{1}{A}\int (1-u^2)\,du = -\frac{1}{A}\left(u - \frac{u^3}{3}\right) + C = -\frac{1}{A}\left(\cos(Ax) - \frac{(\cos(Ax))^3}{3}\right) + C$. The definite integral is not nice: $-\frac{1}{A}\left(\cos(Ax) - \frac{(\cos(Ax))^3}{3}\right)\Big]_0^2 = -\frac{1}{A}\left(\cos(2A) - \frac{(\cos(2A))^3}{3}\right) + \frac{1}{A}\left(1 - \frac{1}{3}\right)$. m_A is the value of the definite integral divided by the interval's length: $\frac{-\frac{1}{A}\left(\cos(2A) - \frac{(\cos(2A))^3}{3}\right) + \frac{1}{A}\left(1-\frac{1}{3}\right)}{2}$.



b) What is $\lim_{A\to\infty} m_A$?

Answer The limit is 0. The limit of $\frac{1}{A}\left(1 - \frac{1}{3}\right)$ is easy. The other part, $-\frac{1}{A}\left(\cos(2A) - \frac{(\cos(2A))^3}{3}\right)$, has limit 0 because the bottom, A, goes to ∞ and the top is bounded since cosine's values vary between -1 and $+1$. To the right is a graph when $A = 30$. Observe that there's much cancellation (area above and below the $x-$axis). More cancellation occurs as A increases. The *net area* is at most one increasingly narrow bump.

(14) 8. Find $\int \frac{1}{x^2\sqrt{x^2-3}}\,dx$.

Answer Try $x = \sqrt{3}\sec(\theta)$. Then $x^2 - 3 = 3(\sec(\theta))^2 - 3 = 3(\tan(\theta))^2$ so $\sqrt{x^2-3} = \sqrt{3}\tan(\theta)$. Also $dx = \sqrt{3}\sec(\theta)\tan(\theta)\,d\theta$. So: $\int \frac{1}{x^2\sqrt{x^2-3}}\,dx = \int \frac{1}{\left(\sqrt{3}\sec(\theta)\right)^2\sqrt{3}\tan(\theta)}\sqrt{3}\sec(\theta)\tan(\theta)\,d\theta = \frac{1}{3}\int \frac{1}{\sec(\theta)}\,d\theta = \frac{1}{3}\int\cos(\theta)\,d\theta = \frac{1}{3}\sin(\theta) + C$. Since $\sec(\theta) = \frac{x}{\sqrt{3}}$, $\cos(\theta) = \frac{\sqrt{3}}{x}$ and $\sin(\theta) = \sqrt{1 - (\cos(\theta))^2} = \sqrt{1 - \left(\frac{\sqrt{3}}{x}\right)^2}$ so the indefinite integral is $\frac{1}{3}\sqrt{1 - \left(\frac{\sqrt{3}}{x}\right)^2} + C$.

Another advertisement, done by a computer in one-fiftieth (.02) of a second:

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> int(1/(x^2*sqrt(x^2-3)),x);
                          2      1/2
                        (x  - 3)
                        ----------
                           3 x
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