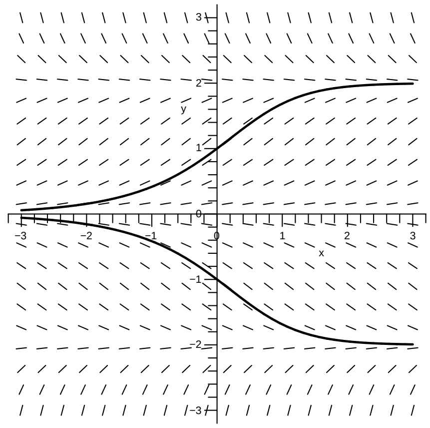
640:152: $\left\{ \begin{matrix} 5, & 6, & 7 \\ 9, & 10, & 11 \end{matrix} \right\}$ **Answers to version A of the Second Exam** 4/10/2007

Here are answers that would earn full credit. Other methods may also be valid.

(12) 1. The graph is a direction field for the differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$.



a) Find the equilibrium solutions (where y doesn't change) for this differential equation. **Answer** $y = 0$ and $y = 2$ and $y = -2$.

b) Sketch solution curves on the graph above through the points below and find the indicated limits:

- $(0, 1)$. Label this curve **A**. On curve **A**, $\lim_{x\to-\infty} y(x) = \underline{\ 0\ }$ and $\lim_{x\to+\infty} y(x) = \underline{\ 2\ }$.
- $(0, -1)$. Label this curve **B**. On curve **B**, $\lim_{x\to-\infty} y(x) = \underline{\ 0\ }$ and $\lim_{x\to+\infty} y(x) = \underline{\ -2\ }$.

c) One of the equilibrium solutions is *not* stable as $x \to +\infty$. Which is that solution? **Answer** $y = 0$.

Comment These solution curves were drawn with numerical approximation methods using `Maple`.

(14) 2. Find the solution of the differential equation $y' = \frac{xy}{\ln y}$ which satisfies the initial condition $y(0) = 3$. In the answer express y explicitly as a function of x.

Answer Separate and integrate: $\int \frac{\ln y}{y} dy = \int x\,dx$ (using the substitution $u = \ln y$ for the left side if needed) so that $\frac{(\ln y)^2}{2} = \frac{x^2}{2} + C$. The initial condition $(0, 3)$ then gives $\frac{(\ln 3)^2}{2} = C$ so $\frac{(\ln y)^2}{2} = \frac{x^2}{2} + \frac{(\ln 3)^2}{2}$ so $(\ln y)^2 = x^2 + (\ln 3)^2$ so $\ln y = \sqrt{x^2 + (\ln 3)^2}$ (the positive square root is needed to get the solution curve through $(0, 3)$ rather than $(0, -3)$!) and then $y = e^{\sqrt{x^2+(\ln 3)^2}}$ is the explicit function of x requested.

(12) 3. The series $\sum_{n=1}^{\infty} \frac{1}{7\sqrt{n}+2^n}$ converges and its sum, to an accuracy of .001, is .314. Find a positive integer N so that the partial sum, $S_N = \sum_{n=1}^{N} \frac{1}{7\sqrt{n}+2^n}$, has a value within .001 of the sum. Explain your reasoning.

Answer The key observation is that $\frac{1}{7\sqrt{n}+2^n} < \frac{1}{2^n}$ which is certainly true when n is a positive integer. Then $0 \le \sum_{n=1}^{\infty} \frac{1}{7\sqrt{n}+2^n} - \sum_{n=1}^{N} \frac{1}{7\sqrt{n}+2^n} = \sum_{n=N+1}^{\infty} \frac{1}{7\sqrt{n}+2^n} < \sum_{n=N+1}^{\infty} \frac{1}{2^n} \overset{\text{geom.}}{\underset{\text{series}}{=}} \frac{\frac{1}{2^{N+1}}}{1-\frac{1}{2}} = \frac{1}{2^N}$. Since $.001 = \frac{1}{1{,}000}$ we can consult the table given and see that $N = 10$ will be enough since $\frac{1}{1{,}024} < \frac{1}{1{,}000}$.

(14) 4. This problem is about the power series $\sum_{n=1}^{\infty} \left(\frac{3n+2}{n^2}\right) x^n$.

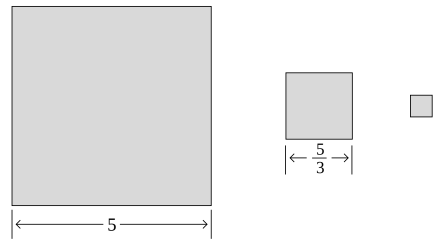
a) What is the radius of convergence of this power series?

Answer $a_n = \left(\frac{3n+2}{n^2}\right) x^n$ so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{3(n+1)+2}{(n+1)^2}\right)x^{n+1}}{\left(\frac{3n+2}{n^2}\right)x^n}\right| = \frac{(3n+5)n^2}{(n+1)^2(3n+2)}|x|$. As $n \to \infty$, this ratio $\to 1$. The radius of convergence therefore must be 1 using the Ratio Test. The Root Test can also be used.

b) What is the behavior of the power series (divergence, absolute or conditional convergence) on the boundary points of the interval of convergence?

Answer If $x = 1$ the series becomes $\sum_{n=1}^{\infty} \left(\frac{3n+2}{n^2}\right)$. This is $\sum_{n=1}^{\infty} \left(\frac{3}{n} + \frac{2}{n^2}\right)$. The first part is three times the harmonic series, so (this is a series of positive terms!) this series diverges. Other "Tests" can also be used. If $x = -1$, the series becomes $\sum_{n=1}^{\infty} \left(\frac{3n+2}{n^2}\right)(-1)^n$. This satisfies all of the conditions of the Alternating Series Test (the signs alternate, the terms without signs decrease, and the limit of the terms is 0). Perhaps the only condition which is not immediate is that the terms decrease. This can be checked with calculus, but certainly (linear growth on top, quadratic growth on the bottom) for n sufficiently large, the terms without signs decrease, and we only care about the infinite tail. This series converges, and since the original series diverges for $x = 1$, this series converges conditionally.

(12) 5. An infinite sequence of squares is drawn. The first three are shown below. The first square has side length 5, and each square after the first square has side length equal to one-third of the preceding square's side length.



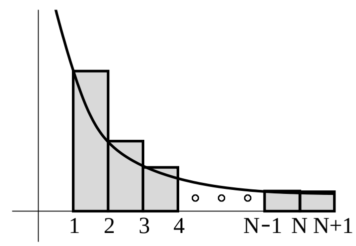
a) What is the total length of the perimeter of all of the squares?

Answer The first square has perimeter equal to $4\cdot 5$. The second square has perimeter equal to $4\cdot\frac{5}{3}$. The third square has perimeter equal to $4\cdot\frac{5}{3^2}$. **Etc.** The total length is the sum of a geometric series with $a=20$ and $r=\frac{1}{3}$. The sum is $\frac{a}{1-r}=\frac{20}{1-\frac{1}{3}}$. This is a fine answer, or you can "simplify" to 30.

b) What is the total area inside all of the squares?

Answer The first square has area equal to 5^2. The second square has area equal to $\left(\frac{5}{3}\right)^2=\frac{5^2}{3^2}$. The third square has area equal to $\left(\frac{5}{3^2}\right)^2=\frac{5^2}{3^4}$. **Etc.** The total area is the sum of a geometric series with $a=5^2=25$ and $r=\frac{1}{3^2}=\frac{1}{9}$. The sum is $\frac{a}{1-r}=\frac{25}{1-\frac{1}{9}}$. This is a fine answer, or you can "simplify" (why?) to $\frac{225}{8}$.

(12) 6. The infinite series $\sum_{n=1}^{\infty}\frac{4}{n^{1/3}}$ diverges. Find N so that the partial sum, $\sum_{n=1}^{N}\frac{4}{n^{1/3}}$, is larger than 100.



Answer Compare the partial sum to an integral. The picture to the right (which is correct since $x^{-1/3}$ is decreasing for $x>0$) verifies the inequality $\sum_{n=1}^{N}\frac{4}{n^{1/3}}>\int_1^{N+1}\frac{4}{x^{1/3}}\,dx$. Now integrate: $\int_1^{N+1}\frac{4}{x^{1/3}}\,dx=4\left(\frac{3}{2}\right)x^{2/3}\Big]_1^{N+1}=6(N+1)^{2/3}-6$. To make this *larger than* 100, I'll take $N=10^6-1=1{,}000{,}000-1=999{,}999$ (many other choices work!). Then $N+1=10^6$ so that $(N+1)^{2/3}=(10^6)^{2/3}=10^4$. Surely $60{,}000-6$ is larger than 100.

(12) 7. a) Suppose the sequence $\{a_n\}$ is defined by $a_n=\left(1+\frac{3}{n}\right)^{2n}$. Find the exact value of the limit of this sequence.

Answer Take logs: $\ln(a_n)=\ln\left(\left(1+\frac{3}{n}\right)^{2n}\right)=2n\ln\left(1+\frac{3}{n}\right)=\frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{2n}}$. As $n\to\infty$, this is $\frac{0}{0}$ so try l'Hospital's rule: $\lim_{n\to\infty}\frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{2n}}=\lim_{n\to\infty}\frac{\frac{d}{dn}\left(\ln\left(1+\frac{3}{n}\right)\right)}{\frac{d}{dn}\left(\frac{1}{2n}\right)}$ and we check if the latter limit exists. It is $\lim_{n\to\infty}\frac{\frac{1}{1+\frac{3}{n}}\cdot\left(-\frac{3}{n^2}\right)}{-\frac{1}{2n^2}}=\lim_{n\to\infty}\frac{6}{1+\frac{3}{n}}=6$. Therefore $\lim_{n\to\infty}a_n=\lim_{n\to\infty}e^{\ln a_n}=e^6$.

This answer can also be obtained by manipulating the limit for e given on the formula sheet.

b) Assume that the numbers b_n are defined recursively by $b_1=1$, $b_{n+1}=\sqrt{5+2b_n}$ for $n=1,2,3,\ldots$. Assume also that we have already shown that the sequence $\{b_n\}$ converges. Find $\lim_{n\to\infty}b_n$.

Answer If $\lim_{n\to\infty}b_n=L$ then $\lim_{n\to\infty}b_{n+1}=L$. Since the various functions involved (square root, sum, etc.) are all continuous, this implies that $L=\sqrt{5+2L}$. Then $L^2-2L-5=$ so that $L=\frac{+2\pm\sqrt{4+20}}{2}=1\pm\sqrt{6}$. I will choose the positive sign since all of the b_n's are positive. So the limit is $1+\sqrt{6}$.

(12) 8. a) Find the power series for $\frac{1}{2+x^3}$ centered at $x=0$. Either write at least the first 4 non-zero terms of the series, or write the whole series in summation form.

Answer $\frac{1}{2+x^3}=\frac{\frac{1}{2}}{1-\left(-\frac{1}{2}x^3\right)}\overset{\text{geom.}}{\underset{\text{series}}{=}}\frac{1}{2}-\frac{1}{4}x^3+\frac{1}{8}x^6-\frac{1}{16}x^9+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{3n}$.

b) Find the first 6 non-zero terms of the power series for $\frac{5-7x}{2+x^3}$ centered at $x=0$. The answer should be a polynomial in x with exactly 6 terms of different degrees in x and with non-zero coefficients.

Answer $(5-7x)\left(\frac{1}{2}-\frac{1}{4}x^3+\frac{1}{8}x^6\right)=5\left(\frac{1}{2}-\frac{1}{4}x^3+\frac{1}{8}x^6\right)-7x\left(\frac{1}{2}-\frac{1}{4}x^3+\frac{1}{8}x^6\right)=\frac{5}{2}-\frac{5}{4}x^3+\frac{5}{8}x^6-\frac{7}{2}x+\frac{7}{4}x^4-\frac{7}{8}x^7$. In a more standard order, this is $\frac{5}{2}-\frac{7}{2}x-\frac{5}{4}x^3+\frac{7}{4}x^4+\frac{5}{8}x^6-\frac{7}{8}x^7$.