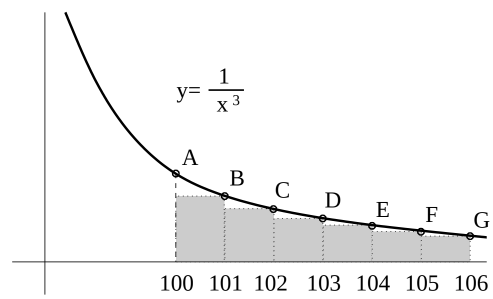
Using an integral to estimate an infinite tail

1. A computer reports the following information:

$$\sum_{j=1}^{10} \frac{1}{j^3+2j^2+j} \approx 0.35105; \qquad \sum_{j=1}^{100} \frac{1}{j^3+2j^2+j} \approx 0.35501; \qquad \sum_{j=1}^{1000} \frac{1}{j^3+2j^2+j} \approx 0.35506.$$

This suggests that $\sum_{j=1}^{\infty} \frac{1}{j^3+2j^2+j}$ converges and that its value (to at least 3 decimal places) is 0.355. The following steps show why this is true. Notice that the series has all positive terms, and therefore if you know that the "infinite tail" $\sum_{j=101}^{\infty} \frac{1}{j^3+2j^2+j}$ converges and has sum less than .001, the omitted terms after the first 100 of the whole series won't matter to 3 decimal places. The numerical evidence above is enough.

Answer Certainly if j is a positive integer, $\frac{1}{j^3+2j^2+j} \leq \frac{1}{j^3}$. The graph of $y = \frac{1}{x^3}$ for $x \geq 100$ is shown to the right. The points A through G on the graph have coordinates $\left(100, \frac{1}{100^3}\right)$, $\left(101, \frac{1}{101^3}\right)$, $\left(102, \frac{1}{102^3}\right)$, $\left(103, \frac{1}{103^3}\right)$, $\left(104, \frac{1}{104^3}\right)$, $\left(105, \frac{1}{105^3}\right)$, and $\left(106, \frac{1}{106^3}\right)$, respectively. The second coordinates of the points B through G are the heights of the boxes. The picture explains why the improper integral $\int_{100}^{\infty} \frac{1}{x^3}\,dx$ overestimates $\sum_{j=101}^{\infty} \frac{1}{j^3}$. We can get an overestimate of the infinite tail of this simpler series (and therefore an overestimate of the original infinite tail) by computing the improper integral. Here's the computation: $\int_{100}^{\infty} \frac{1}{x^3}\,dx = \lim_{A\to\infty} \int_{100}^{A} \frac{1}{x^3}\,dx = \lim_{A\to\infty} -\frac{1}{2x^2}\Big]_{100}^{A} = \lim_{A\to\infty} -\frac{1}{2A^2} + \frac{1}{2\cdot 100^2} = \frac{1}{2\cdot 100^2} = 0.00005$. Since this is less than .001, we're done.



Note Certainly $\frac{1}{j^3+2j^2+j} \leq \frac{1}{2j^2}$ also. If this estimate is used, then the corresponding improper integral is $\int_{100}^{\infty} \frac{1}{2x^2}\,dx$ which is equal to 0.005, and this is not "good enough" to be less than 0.001. The $\frac{1}{x^3}$ integral gives more information because $\frac{1}{x^3}$ decreases faster than $\frac{1}{2x^2}$.

Actually, the function $\frac{1}{x^3+2x^2+x}$ can be antidifferentiated using partial fractions. The antiderivative is $-\ln(x+1) + \frac{1}{1+x} + \ln x$. The value of the improper integral $\int_{100}^{\infty} \frac{1}{x^3+2x^2+x}\,dx$ is about 0.000049, which is certainly good enough. But that's more work than I would want to do for this problem.

OVER

Using a geometric series to estimate an infinite tail

2. A computer reports the following information:

$$\sum_{j=1}^{10} \frac{30^j}{(j!)^2} \approx 6963.86479; \qquad \sum_{j=1}^{15} \frac{30^j}{(j!)^2} \approx 6977.78140; \qquad \sum_{j=1}^{20} \frac{30^j}{(j!)^2} \approx 6977.78249.$$

This suggests that $\sum_{j=1}^{\infty} \frac{30^j}{(j!)^2}$ converges and that its value (to at least 2 decimal places) is 6977.78.

The following steps show why this is true. Notice that the series has all positive terms, and therefore if you know that the "infinite tail" $\sum_{j=16}^{\infty} \frac{30^j}{(j!)^2}$ converges and has sum less than .002, the omitted terms after the first 15 of the whole series won't matter to 2 decimal places. The numerical evidence above is enough. In what follows, $a_j = \frac{30^j}{(j!)^2}$.

a) If j is a positive integer, simplify the algebraic expression $\frac{a_{j+1}}{a_j}$. The result won't be complicated.

Answer $$\frac{a_{j+1}}{a_j} = \frac{\frac{30^{j+1}}{((j+1)!)^2}}{\frac{30^j}{(j!)^2}} = \frac{\overbrace{30^{j+1}}(j!)^2}{30^j\left(\overbrace{(j+1)!}\right)^2} = \frac{\overbrace{30 \cdot 30^j}(j!)^2}{30^j\overbrace{\left((j+1)j!\right)^2}} = \frac{30 \cdot 30^j (j!)^2}{30^j\,\overbrace{(j+1)^2\,(j!)^2}} = \frac{30}{(j+1)^2}.$$

To do this, you must know (in order, for each $\overbrace{\text{ITEM SHOWN}}$) that $30^{j+1} = 30 \cdot 30^j$ and that $(j+1)! = (j+1)j!$ and that $(\alpha\beta)^2 = (\alpha)^2(\beta)^2$.

b) Use the result from a) to show that if $j \geq 16$, then $\frac{a_{j+1}}{a_j} < 0.11$. (Show all steps. You'll need a calculator!)

Answer If $j \geq 16$, then $j+1 \geq 17$ and $(j+1)^2 \geq 17^2 = 289$. Therefore $\frac{1}{(j+1)^2} \leq \frac{1}{289}$. Then $\frac{a_{j+1}}{a_j}$, which is less than $\frac{30}{(j+1)^2}$ by a), must be less than $\frac{30}{289} \approx 0.10381$. That's certainly less than 0.11.

c) You may assume (this is true!) that $a_{16} \approx .000983$. Use this fact and what was done in b) to compare $\sum_{j=16}^{\infty} a_j$ to a geometric series, each of whose terms is individually larger than this series. Find the sum of the geometric series, which should be less than .002, so that the omitted infinite tail of the original series is small enough. (Show all steps. You'll need a calculator!)

Answer The result of b) implies that $a_{j+1} < a_j(0.11)$ for $j \geq 16$. We will use this repeatedly.

$$\sum_{j=16}^{\infty} a_j = \underline{a_{16}} + a_{17} + a_{18} + a_{19} + a_{20} + \ldots <$$

$$\underline{a_{16} + a_{16}(0.11)} + a_{17}(0.11) + a_{18}(0.11) + a_{19}(0.11) + \ldots <$$

$$\underline{a_{16} + a_{16}(0.11) + a_{16}(0.11)^2} + a_{17}(0.11)^2 + a_{18}(0.11)^2 + \ldots <$$

$$\underline{a_{16} + a_{16}(0.11) + a_{16}(0.11)^2 + a_{16}(0.11)^3} + a_{17}(0.11)^3 + \ldots <$$

$$\underline{a_{16} + a_{16}(0.11) + a_{16}(0.11)^2 + a_{16}(0.11)^3 + a_{16}(0.11)^4} + \ldots <$$

Here the *strategy* is to push down the subscript in each term until the subscript is 16 and compensate for each "push" with a multiplication by 0.11. The work in b) guarantees that this trade only increases the sum.

So we have overestimated the original infinite tail by a geometric series whose first term is a_{16} and whose ratio is 0.11. But we know that $a_{16} \approx .000983$ so the sum of the geometric series which overestimates the infinite tail is $\frac{\text{FIRST TERM}}{1-\text{RATIO}}$ and this is $\frac{.000983}{1-0.11} \approx 0.0011$ which is certainly less than .002.