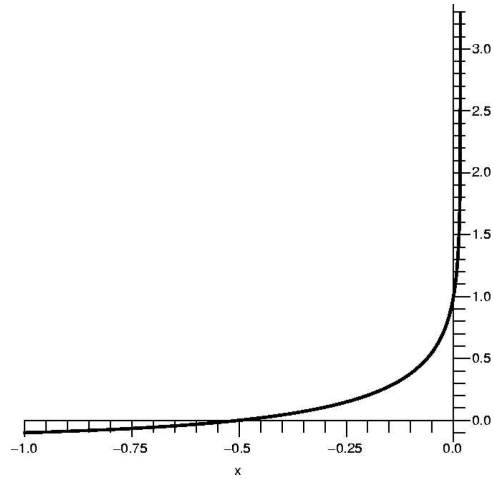
Separable Differential Equations

1. Consider the differential equations

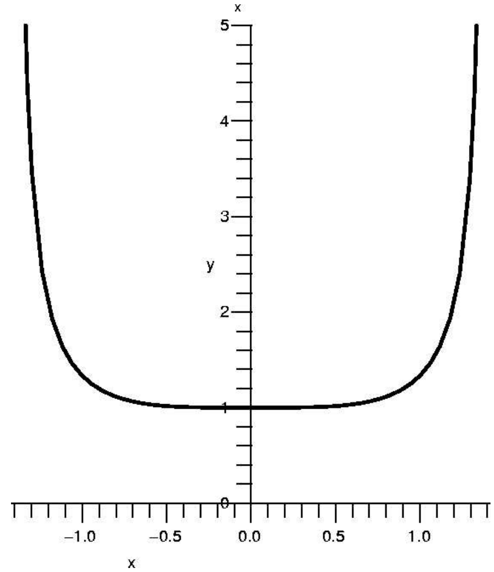
a) $\dfrac{dy}{dx} = 2x + 3y$ b) $\dfrac{dy}{dx} = e^{2x+3y}$ c) $\dfrac{dy}{dx} = x^3y^2$ d) $\dfrac{dy}{dx} = x^2 + y^3$

Two of these are separable. For each of these two separable equations, solve the initial value problem whose initial condition is $y(0) = 1$. In each case your solution should be $y = f(x)$ where $f(x)$ is a formula.

Answer b) is separable: $e^{2x+3y} = e^{2x}e^{3y}$ so $\int e^{-3y}dy = \int e^{2x}dx$ and $-\frac{1}{3}e^{-3y} = \frac{1}{2}e^{2x} + C$. The initial condition, $(0,1)$, gives $-\frac{1}{3}e^{-3} = \frac{1}{2} + C$ since $e^0 = 1$, and $C = -\frac{1}{3}e^{-3} - \frac{1}{2}$. Therefore $-\frac{1}{3}e^{-3y} = \frac{1}{2}e^{2x} - \frac{1}{3}e^{-3} - \frac{1}{2}$. The solution, $y = f(x)$ is $f(x) = \ln\left(-\frac{3}{2}e^{2x} + e^{-3} + \frac{3}{2}\right)$. The domain: $-\frac{3}{2}e^{2x} + e^{-3} + \frac{3}{2} > 0$ which is $-\frac{3}{2}e^{2x} > -e^{-3} - \frac{3}{2}$ which is $e^{2x} < \frac{2}{3}e^{-3} + 1$ which is $2x < \ln\left(\frac{2}{3}e^{-3} + 1\right)$ which is $x < \ln\left(\sqrt{\frac{2}{3}e^{-3} + 1}\right)$ which is approximately $(-\infty, .016326)$. The graph to the right shows the solution curve on the interval $[-1, .016]$. The solution curve has a vertical asymptote at the right-hand endpoint of its domain.



c) is also separable: That equation becomes $\int y^{-2}dy = \int x^3dx$ and then $-\frac{1}{y} = \frac{1}{4}x^4 + C$. The initial condition, $(0,1)$, yields $-1 = C$. Therefore $-\frac{1}{y} = \frac{1}{4}x^4 - 1$ so $y = f(x)$ where $f(x) = \frac{1}{1-\frac{1}{4}x^4} = \frac{4}{4-x^4}$. The "natural domain" of this formula is all numbers *except* $\pm 4^{1/4}$ but the domain of the solution curve is only $(-4^{1/4}, +4^{1/4})$: these are the only numbers which can get information (?) from the initial condition, because the formula could just as well select a different C in the other intervals. The graph to the right shows the solution curve on $(-4^{1/4}, +4^{1/4})$ but chopped off at $y = 5$ so the graph is for x's in $[-1.337, 1.337]$. The solution curve has vertical asymptotes at both endpoints of its domain.



Sequences

2. Write decimal approximations for the first 5 terms of the sequence $a_n = \frac{20^n}{n!}$, beginning with $n = 1$. It is true that $\lim_{n\to\infty} a_n = 0$. Briefly explain why. (Suggestion: think about how the terms change when n is larger than 40 – do they grow or shrink? How much?)

Answer The first 5 terms are 20., 200.0000000, 1333.333333, 6666.666667, 26666.66667. They certainly *seem* to grow, but consider what happens after a_{40}: the next term, a_{41} is gotten from a_{40} by multiplying. There's a 20 on top and 41 on the bottom because $41! = (41)40!$. The factor $\frac{20}{41}$ is certainly less than $\frac{1}{2}$. All of the factors relating a_{n+1} to a_n for $n \geq 40$ are less than $\frac{1}{2}$. So as $n \to \infty$, a_{40} is being multiplied by more and more factors all less than $\frac{1}{2}$ to get the a_n's. Therefore I think that $0 < a_{40+n} < \left(\frac{1}{2}\right)^n a_{40} \to 0$.

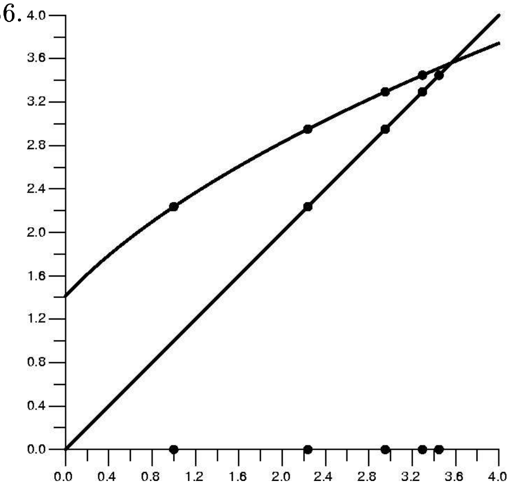
Note Actually $a_{40} \approx 13475.80626$ and the largest a_n is $a_{20} \approx 4.309980412 \cdot 10^7$. But we don't need to know any of these numbers to conclude that the sequence has limit equal to 0.

3. Suppose $f(x) = \sqrt{2+3x}$, and that a sequence $\{a_n\}$ is defined by the following recursive process:

$a_1 = 1$; $a_{n+1} = f(a_n)$ for $n > 1$.

a) Compute decimal approximations for the first 5 terms, a_1, a_2, a_3, a_4, and a_5, of the sequence.

Answer 1., 2.236067977, 2.950966610, 3.294373966, 3.447190436.



b) The graph to the right shows parts of the line $y = x$ and the curve $y = \sqrt{2+3x}$. Locate on this graph or on a copy to be handed in the following points: (a_1, a_2), (a_2, a_2), (a_2, a_3), (a_3, a_3), (a_3, a_4), (a_4, a_4), (a_4, a_5), and (a_5, a_5). Also show a_1, a_2, a_3, a_4, and a_5 on the $x-$axis. (You must draw **13 points**.)

Answer The dots are shown on the graph to the right. Of course if the dots are drawn in the order specified, maybe the relationship of the sequence to the geometry can be understood. Start at $x = 1$ on the $x-$axis, go up to the curve, "bounce" horizontally to the diagonal line, then up to the curve again, then bounce, etc. The dots on the $x-$axis are below these dots, and they are the sequence $\{a_n\}$.

c) Write a statement of a result from section 11.1 which shows that this sequence converges.

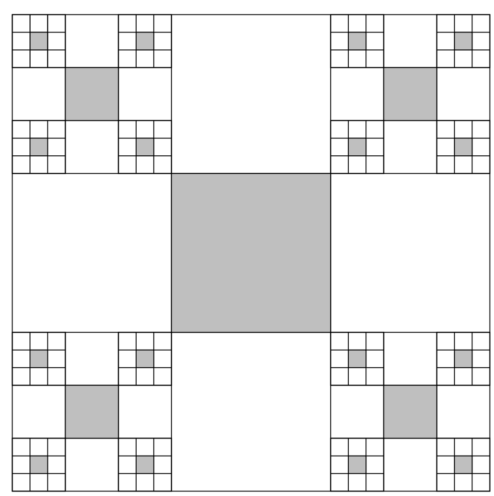
Answer "Every bounded, monotonic sequence is convergent." This is on p. 709. The sequence is increasing, and is certainly bounded (it is squashed inside the curve and the line, below the intersection point).

d) Compute the limit of $\{a_n\}$.

Answer The sequence converges to the value of x for which $x = \sqrt{2+3x}$ or $x^2 = 2+3x$ or $x^2-3x-2 = 0$. The roots of this equation are $\frac{+3\pm\sqrt{3^2-4(-2)(1)}}{2} = \frac{+3\pm\sqrt{17}}{2}$. The limit is the positive root, $\frac{3+\sqrt{17}}{2} \approx 3.561552813$.

Series

4. A 1×1 square is "dissected" by three equally spaced horizontal lines and by three equally spaced vertical lines. The central square is shaded. Then the bordering Northeast, Northwest, Southeast, and Southwest squares are similarly dissected, with the central square shaded. Each of *those* dissected squares has something similarly done to their borders, etc. The diagram to the right shows this process only for the first three steps.



a) How many new shaded squares are introduced at the n^{th} step? (There is one shaded square at the first step.) What is the side length of the squares which are introduced at the n^{th} step?

Answer There are 4 squares at the second step, and 16 squares at the third step. So there are 4^{n-1} squares at the n^{th} step. The first square has side length $\frac{1}{3}$. The second step squares have side length equal to one-third of that, which is $\frac{1}{3^2}$. I think that the side length of the squares belonging to the n^{th} step is $\frac{1}{3^n}$.

b) What is the total sum, as n goes from 1 to ∞, of the shaded area (all the shaded squares)?

Answer The area of a square belonging to the n^{th} step is the square of its side length, so the area is $\left(\frac{1}{3^n}\right)^2 = \frac{1}{3^{2n}}$. Since there are 4^{n-1} squares at this step, the total area of these squares is $\frac{4^{n-1}}{3^{2n}}$. If you find this difficult to understand, please write out the first few steps:

$$\frac{1}{9} + \frac{4}{81} + \frac{16}{729} + \ldots = \frac{1}{3^2} + \frac{4}{3^4} + \frac{4^2}{3^6} + \frac{4^3}{3^8} + \ldots$$

This is a *geometric series* whose first term is $a = \frac{1}{9}$, which has constant ratio between successive terms $r = \frac{4}{9}$. The sum of this series is $\frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{4}{9}} = \frac{1}{5}$.