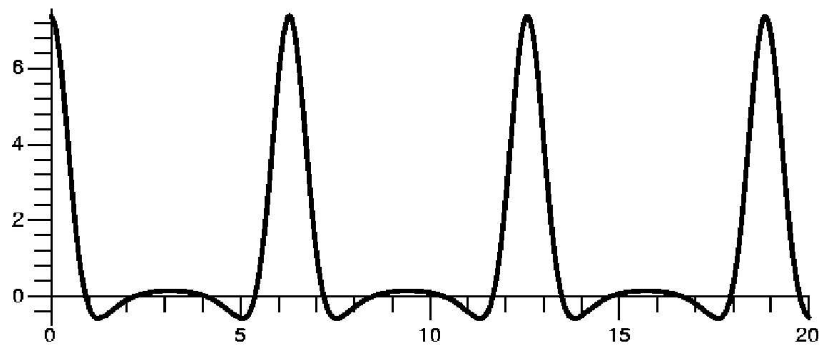
An answer to problem 4 on workshop 5 for 152:05–07 and 09–11 4/20/2007

4. Define $f(x)$ with the sum $f(x) = \sum\limits_{n=0}^{\infty} \frac{2^n \cos(nx)}{n!}$. This is <u>not</u> a power series. Below is a graph of the partial sum $s_{100}(x) = \sum\limits_{n=0}^{100} \frac{2^n \cos(nx)}{n!}$ of the series for $0 \le x \le 20$.



a) Verify that the series defining $f(x)$ converges for all x.

Answer We will prove that the series converges absolutely for all x and because absolute convergence implies convergence, the series converges for all x. We know $|\cos(\text{ANYTHING})| \le 1$ so $\left|\frac{2^n \cos(nx)}{n!}\right| \le \frac{2^n}{n!}$. The series which has n^{th} term equal to $\frac{2^n}{n!}$ converges by the Ratio Test: the ratio between successive terms is $\frac{2}{n+1}$ and this $\to 0$ as $n \to \infty$, and $0 < 1$.

b) Is the apparent periodicity of the function $f(x)$ actually correct? If yes, explain why.

Answer Yes, $f(x)$ is periodic with period 2π. For integer n, $\cos(n(x + 2\pi)) = \cos(nx + 2n\pi) = \cos(nx)$ since cosine is 2π periodic. So all the terms in the infinite series for $f(x + 2\pi)$ are identical to the terms in the infinite series for $f(x)$.

c) Verify that the actual graph of the function is always within .01 of the graph shown. That is, if x is any real number, then $|f(x) - s_{100}(x)| < .01$.

Answer Again the observation $|\cos(\text{ANYTHING})| \le 1$ will simplify our work. Therefore $|f(x) - s_{100}(x)| = \left|\sum\limits_{n=0}^{\infty} \frac{2^n \cos(nx)}{n!} - \sum\limits_{n=0}^{100} \frac{2^n \cos(nx)}{n!}\right| = \left|\sum\limits_{n=101}^{\infty} \frac{2^n \cos(nx)}{n!}\right| \le \sum\limits_{n=101}^{\infty} \frac{2^n}{n!}$. This infinite tail can be overestimated by a geometric series because the ratio between successive terms of the tail series is $\frac{2}{n+1}$ (the tops of the tail series terms are powers of 2, and the bottom are factorials). Here $n \ge 101$ so the ratio is at most $\frac{2}{102} = \frac{1}{51}$. So the tail series is less than the geometric series with $a = \frac{2^{101}}{(101)!}$ and $r = \frac{1}{51}$. This is $\sum\limits_{n=0}^{\infty} \frac{2^{101}}{(101)!}\left(\frac{1}{51}\right)^n = \frac{\frac{2^{101}}{(101)!}}{1-\left(\frac{1}{51}\right)}$.

Using the numbers supplied, the (approximate) value is $\frac{\frac{2.54 \cdot 10^{30}}{9.43 \cdot 10^{159}}}{.98} = .27 \cdot 10^{-129}$. This result is much smaller than .001 so the graph shown is very much like the true graph.

A discussion of how to write a simple formula for the sum of this series is on the next page.

Comment I discussed Euler's formula *very* briefly in class on Wednesday, April 18. You may not "believe" in the formula, but here is one result which follows from it. Although maybe each step is almost easy, I do *not* claim that the whole journey is obvious, and certainly the final result is *not* obvious.

Step 1 Euler's formula states that $e^{ix} = \cos x + i\sin x$.

Step 2 If we substitute nx for x, we see that $e^{inx} = \cos(nx) + i\sin(nx)$.

Step 3 This problem is about the series $\sum_{n=0}^{\infty} \frac{2^n\cos(nx)}{n!}$. But the preceding step makes me want to consider the following: $\left(\sum_{n=0}^{\infty} \frac{2^n\cos(nx)}{n!}\right) + i\left(\sum_{n=0}^{\infty} \frac{2^n\sin(nx)}{n!}\right)$.

Step 4 So we are looking at $\sum_{n=0}^{\infty} \frac{2^n(\cos(nx)+i\sin(nx))}{n!}$ which is equal to $\sum_{n=0}^{\infty} \frac{2^n\left(e^{inx}\right)}{n!}$.

Step 5 But $e^{inx} = \left(e^{ix}\right)^n$, so this is the series $\sum_{n=0}^{\infty} \frac{2^n\left(e^{ix}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(2e^{ix}\right)^n}{n!}$.

Step 6 The exponential function *is* $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. The series we are considering has $A = 2e^{ix}$, so the sum of the series must be $e^{\left(2e^{ix}\right)}$.

Step 6 Euler's formula states that this is $e^{2(\cos x + i\sin x)} = e^{2\cos x + 2i\sin x}$.

Step 7 The exponential function converts addition to multiplication and therefore we know $e^{2\cos x+2i\sin x} = e^{2\cos x}e^{2i\sin x}$.

Step 8 Look at $e^{2i\sin x} = e^{i(2\sin x)}$. Use Euler's formula again, replacing the x in the original formula with $2\sin x$. The result is $e^{i(2\sin x)} = \cos(2\sin x) + i\sin(2\sin x)$.

Step 9 The sum of the series is $e^{2\cos x}e^{i(2\sin x)} = e^{2\cos x}\left(\cos(2\sin x) + i\sin(2\sin x)\right) = e^{2\cos x}\cos(2\sin x) + ie^{2\cos x}\sin(2\sin x)$.

Step 10 Compare the results of **Step 3** and the preceding step. The same quantities are being described. The "real parts" (the things without the i) should be the same, so therefore (not "clearly", definitely not "clearly"!):

$$\sum_{n=0}^{\infty} \frac{2^n\cos(nx)}{n!} = e^{2\cos x}\cos(2\sin x)$$