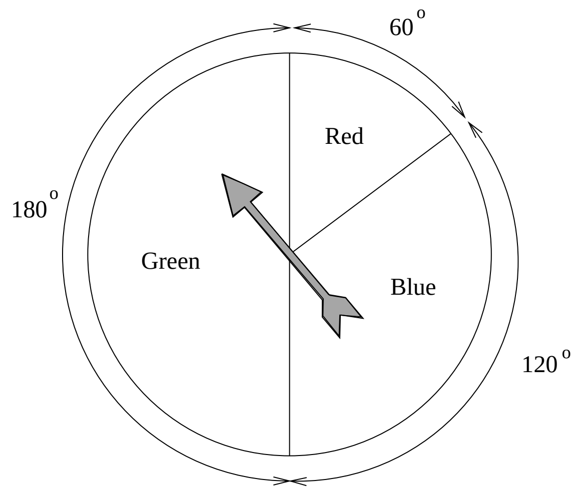
Starting example

Suppose that we have a spinner as shown to the right. What is the chance that an "honest" or "fair" spin lands on **Green**? I think it is $\frac{180°}{360°}$, or $\frac{1}{2}$. Similarly, we can easily see that the chance of **Blue** is $\frac{120°}{360°}$, or $\frac{1}{3}$, while the chance of **Red** is $\frac{1}{6}$. This is easy. So:

$$P(\textbf{Green}) = \tfrac{1}{2};\ P(\textbf{Blue}) = \tfrac{1}{3};\ P(\textbf{Red}) = \tfrac{1}{6}.$$

Here P(something) is the probability or chance that "something" happens. Certainly the probability should be between 0 and 1 and the sum of all the "somethings" that happen should be 1.



What may not be apparent is how much you *win* if each color pays a different amount. For example, suppose that

$$\begin{cases} \text{a } \textbf{Green} \text{ spin pays off \$30} \\ \text{a } \textbf{Blue} \text{ spin pays off \$15} \\ \text{a } \textbf{Red} \text{ spin pays off \$75} \end{cases}$$

How much is an "average" spin worth?

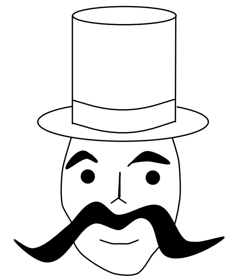
Phrased a bit differently, how much should someone be willing to *pay* to play this game? One naive answer might be: since there are three possible outcomes, and three possible rewards, then the average reward of a spin is the average of the three outcomes, or \$ $\frac{30+15+75}{3}$ = \$40. Some consideration of extreme cases might show that's too simple. If **Green** were worth \$10,000 and the other two colors were worth nothing, then we'd expect about half of our spins to be **Green** in the long run, and about half the time to win \$10,000 per spin, so that the *average* winning per spin would be \$5,000. In our 30–15–75 payoff plan, we must compute a *weighted* average, and the weights are the chances, the probabilities, that each color will occur. So arrange things:

Outcome	**Probability**	**Payoff per spin**	**Expected winnings**
Green	$\frac{1}{2}$	\$30	\$15.00
Blue	$\frac{1}{3}$	\$15	\$5.00
Red	$\frac{1}{6}$	\$75	\$12.50

The total expected winnings (the *expectation*) will be \$32.50, the correct weighted average of the probabilities and the payoffs. An entry "fee" of less than \$32.50 would, in the long run, over many plays, yield a profit to the player. An entry fee greater than \$32.50 would, in the long run, over many plays, profit the "proprietor" of the spinner.

A coin flipping game

Let's try some real gambling. A gentleman just off the riverboat comes up to you, smiling. He carries a fair coin, with two different sides, one side heads, **H**, and one side tails, **T**. He offers to play a game with you. He will toss the coin. If **H** appears, he will pay you \$1. If **T** appears, he will toss the coin again. If, on the second toss, **H** appears, he will pay you \$2. If **T** appears again, he will toss another time. And so on ....


Your friendly gambler

In this case the "game" is complicated, and I think it needs to be specified more fully. The set of outcomes is the collection of coin tosses, **T T T . . . T H**. That is, for each non-negative integer, n, one possible outcome is $(n-1)$ **T**'s followed by an **H**. Let's call this $\mathcal{S}_n$. What's the probability of $\mathcal{S}_n$? We're asking for n straight specified tosses of a fair coin, so $P(\mathcal{S}_n)$ must be $\left(\frac{1}{2}\right)^n$. The gambler will pay n dollars if $\mathcal{S}_n$ occurs. Here are some questions to consider:

- What if the coin *never* lands heads?

  The probability of this happening seems to be $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$. This may be an example of a conceivable (?) event which never happens (??).
- Would you pay the gambler, an honest, genial individual, 50 cents to play this game?

  Of course. You've got to win at least a dollar.
- Would you pay the gambler, an honest, genial individual, one dollar to play this game?

  Surely, for the same reason as the previous question.
- Would you pay the gambler, an honest, genial individual, one million dollars to play this game?

  The gambler asserts that there are many, many positive integers. There are only finitely many of these less than one million, and infinitely many which are more than one million. Thus (according to the gambler) there has infinitely *more* chances of paying more than one million than you have of losing one million. (!!!) So pay the million and play the game.

We can try to compute this game's average payoff (its "expectation") in a way that's similar to the spinner game.

Outcome	Probability	Payoff per spin	Expected winnings
H	$\frac{1}{2}$	\$1	\$0.50
T H	$\frac{1}{4}$	\$2	\$0.50
T T H	$\frac{1}{8}$	\$3	\$0.375
. . . . .	...	...	...
$\mathcal{S}_n$	$\frac{1}{2^n}$	\$$n$	\$$\frac{n}{2^n}$
. . . . .	...	...	...

The sum of the probabilities of the outcomes (an infinite list!) is

$$\sum_{n=1}^{\infty} P(\mathcal{S}_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Although we have an "infinite series", it is the friendliest type, a geometric series. If S is the sum of this series, then $S = \frac{a}{1-r}$. Our specific geometric series has both a and r equal to $\frac{1}{2}$, so its sum is 1, as we should hope, since we've made a list of all (non-negligible) possible outcomes.

What about the average payoff? If the entry fee is *less* than the average payoff, in the long run, over repeated plays, we'd expect to profit, and the honest gambler tossing the coin for us would expect to lose money. If the entry fee is *more* than the average payoff, in the long run, over repeated plays, we'd expect to loose, and our honest, genial friend would expect to profit.

We need to look at the sum of the expected winnings:

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots = \sum_{n=1}^{\infty} \frac{n}{2^n}$$

It is perfectly possible that this expectation *could* be infinite. To emphasize this, modify the game to one in which the gambler kindly offers to pay out 2^n dollars if the first head occurs on the n^{th} toss: e.g., outcome $\mathcal{S}_n$. Then every play of the game offers us a chance to win (on average) $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$ dollars. This seems to be quite a lot! But what does it mean? There is no one play or coin toss sequence which will win infinitely many dollars. It really means that there's no upper bound on the average amount of winnings in the changed game so there is no fair entry fee that the gambler could charge – the gambler would always lose money in the long run.

Let's return to the original game, where the payoff is $\sum_{n=1}^{\infty} \frac{n}{2^n}$. This series is not a geometric series since the ratio between successive terms changes. For example, the ratio between the first and second terms is 1, and the ratio between the second and third terms is $\frac{3}{4}$. Let's see how to compute the payoff.

Computing the payoff with magical calculus

"Specialize" the geometric series formula again: change a to 1 and r, to x:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

We can differentiate this formula to get:

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \ldots$$

Now multiply by x:

$$\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

If we fix x to be $\frac{1}{2}$ we obtain:

$$2 = 0 + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \ldots$$

The payoff is \$2.

Another game, and maybe some more magic

Suppose that we're offered a payoff of n^2 for the coin-flipping sequence, $\mathcal{S}_n$. For example, if the first head occurs on the third toss, we receive \$9. The average payoff for this game is then $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. We can compute this by using almost the same tricks:

Begin with

$$\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

Differentiate to get (I used the quotient rule on the left-hand side to get the result shown):

$$\frac{1+x}{(1-x)^3} = 0 + 1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots$$

Multiply by x:

$$\frac{x+x^2}{(1-x)^3} = 0 + 1x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \ldots$$

If we now fix x to be $\frac{1}{2}$ we see the formula:

$$6 = 0 + 1\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{2}\right)^2 + 3^2\left(\frac{1}{2}\right)^3 + 4^2\left(\frac{1}{2}\right)^4 + \ldots$$

The payoff is \$6.