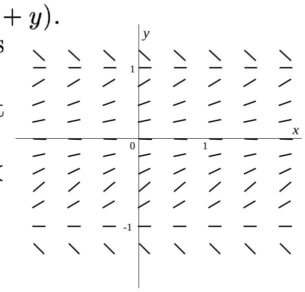
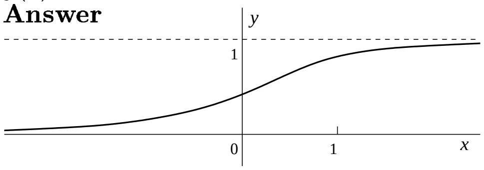
152:5–7 & 9–11 Some answers to *my* review questions for the second exam 4/2/2007

1. Find the solution of the differential equation $\frac{dy}{dx} = \frac{xy^3}{x^2+1}$ satisfying the initial condition $y(0) = 3$. In the answer express y explicitly as a function of x.
Answer This is a separable equation, so we get $\frac{dy}{y^3} = \frac{x\,dx}{x^2+1}$ which can be integrated to $-\frac{1}{2y^2} = (\frac{1}{2}\ln(x^2+1)) + C$. The initial condition $(0,3)$ gives the equation $-\frac{1}{18} = (\ln 1) + C$ so $-\frac{1}{2y^2} = (\frac{1}{2}\ln(x^2+1)) - \frac{1}{18}$ and, solving for y: $y = \frac{1}{\sqrt{\frac{1}{9} - \ln(x^2+1)}}$.

(12) 1. Below is part of the direction field for the differential equation $y' = y^2(1-y)(1+y)$.



a) Find all numbers k so that the constant function $f(x) = k$ is a solution of this differential equation (these are the *equilibrium solutions*).
Answer A constant function has derivative 0. Those $y = k$'s for which y' must always be 0 are the solutions of $y^2(1-y)(1+y) = 0$: k must be 0 or 1 or -1.
b) Sketch a typical solution curve $y = f(x)$ to this differential equation when $0 < y(0) < 1$.
Answer



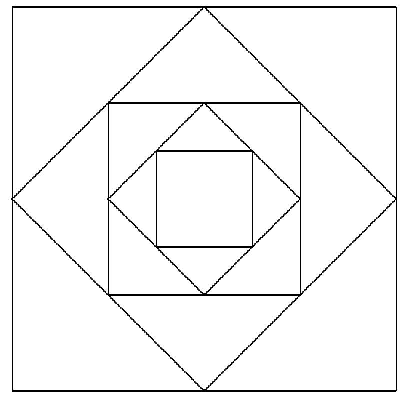
The curve sketched should be located entirely within the strip $0 < y < 1$ and should be increasing with these asymptotic properties evident: $\lim_{x\to+\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 0$.

3. Find the interval of convergence and the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{3^n x^n}{\sqrt{n}}$. In addition, determine whether the series is absolutely or conditionally convergent at the boundary points of the interval of convergence.
Answer If $a_n = \frac{3^n x^n}{\sqrt{n}}$ then $a_{n+1} = \frac{3^{(n+1)} x^{(n+1)}}{\sqrt{n+1}}$ and $\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{(n+1)}|x|^{(n+1)}\sqrt{n}}{3^n|x|^n\sqrt{n+1}} = 3|x| \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$. The limit of this as $n \to \infty$ is $3|x|$ so, using the Ratio Test, the series converges when $|x| < \frac{1}{3}$ and diverges when $x > \frac{1}{3}$. When $x = \frac{1}{3}$ the series is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a p-series with $p = \frac{1}{2} < 1$ which diverges. When $x = -\frac{1}{3}$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, an alternating series which converges conditionally and not absolutely. The interval of convergence is $-\frac{1}{3} \le x < \frac{1}{3}$ and the radius of convergence is $\frac{1}{3}$.

4. The series $\sum_{n=1}^{\infty} \frac{1}{3n^2+5n+7}$ converges. Find a specific finite sum of rational numbers (quotients of integers) which is within .0001 of the sum of the infinite series. Be sure to explain why your error estimate is correct.
Hint Compare the "infinite tail" to something simpler, and analyze that.
Answer Since $\frac{1}{3n^2+5n+7} < \frac{1}{n^2}$, $\sum_{n=N+1}^{\infty} \frac{1}{3n^2+5n+7} < \sum_{n=N+1}^{\infty} \frac{1}{n^2}$. But $\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \int_N^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_N^{\infty} = \frac{1}{N}$. If $N = 10{,}000$ the infinite tail $\sum_{n=N+1}^{\infty} \frac{1}{3n^2+5n+7} < .0001 = \frac{1}{10{,}000}$. The specific finite sum is $\sum_{n=1}^{10{,}000} \frac{1}{3n^2+5n+7}$.

5. a) Suppose the sequence $\{A_n\}$ is defined by $A_n = \frac{6\cdot 4^n + 7n^3}{5\cdot 4^n + 8n^2}$. What is $\lim_{n\to\infty} A_n$?
Answer $A_n = \frac{6\cdot 4^n + 7n^3}{5\cdot 4^n + 8n^2} = \frac{6 + 7\frac{n^3}{4^n}}{5 + 8\frac{n^2}{4^n}}$. But $\lim_{n\to\infty} \frac{n^3}{4^n} = 0$ and $\lim_{n\to\infty} \frac{n^2}{4^n} = 0$ (exponential growth is more rapid than polynomial growth), so $\lim_{n\to\infty} A_n = \frac{6}{5}$.
b) Suppose the sequence $\{B_n\}$ is defined by $B_n = \left(1 + \frac{3}{n}\right)^{5n}$. What is $\lim_{n\to\infty} B_n$? **Hint** ln and l'H.
Answer Take logs: $\ln(B_n) = \ln\left(\left(1+\frac{3}{n}\right)^{5n}\right) = 5n\ln\left(1+\frac{3}{n}\right) = \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{5n}}$. As $n \to \infty$, this is $\frac{0}{0}$ so try l'Hospital's rule: $\lim_{n\to\infty} \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{5n}} = \lim_{n\to\infty} \frac{\frac{d}{dn}\left(\ln\left(1+\frac{3}{n}\right)\right)}{\frac{d}{dn}\left(\frac{1}{5n}\right)}$ and we check if the latter limit exists. It is $\lim_{n\to\infty} \frac{\frac{1}{1+\frac{3}{n}} \cdot -\frac{3}{n^2}}{-\frac{1}{5n^2}} = \lim_{n\to\infty} \frac{15}{1+\frac{3}{n}} = 15$. Therefore $\lim_{n\to\infty} B_n = \lim_{n\to\infty} e^{\ln B_n} = e^{15}$.

6. a) What is the Taylor series for $f(x) = 3 - 11x^5$ centered at $a = 0$? Why?
Answer All the derivatives after the fifth are 0. Therefore the Taylor series *is* the function $f(x)$ itself.
b) Explain briefly why $g(x) = |x|$ has no Taylor series centered at $a = 0$.
Answer g is not differentiable at 0, and therefore $g'(0)$ does not exist.



7. An infinite sequence of squares is drawn (the first five are shown), with the midpoints of the sides of one being the vertices of the next. The outermost square has sides which are 1 unit long. What is the <u>sum</u> of the <u>perimeters</u> of all of the squares?
Answer Two halves of each side of a square get replaced by a hypoteneuse; that is, $\frac{1}{2}S + \frac{1}{2}S \longrightarrow \sqrt{2}\frac{1}{2}S$. But this occurs four times. So $4S \longrightarrow 2\sqrt{2}S$ as we go from the perimeter of one square to the next. We begin with the biggest square which has perimeter 4 ($S = 1$) and see that we must sum a geometric series whose first term is 4 and whose ratio between successive terms is $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$. Thus the sum of the perimeters is $\frac{a}{1-r} = \frac{4}{1-\left(\frac{1}{\sqrt{2}}\right)}$.

8. Determine whether each of the following sequences converges, and find the limits when they exist. Explain your answers.
a) $a_n = \frac{\ln\left(\ln\left(n^4\right)\right)}{(\ln\ n)^3}$
Answer If you "plug in" $n = \infty$. the result is $\frac{\infty}{\infty}$. So probably we should use l'H. I'll begin with some algebra.

$$\lim_{n\to\infty} \frac{\ln\left(\ln\left(n^4\right)\right)}{(\ln\ n)^3} = \lim_{n\to\infty} \frac{\ln(4\ln n)}{(\ln n)^3} = \lim_{n\to\infty} \frac{\ln(4)+\ln(\ln n)}{(\ln n)^3} \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{\frac{1}{\ln n}\cdot\frac{1}{n}}{3(\ln n)^2\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{\ln n}}{3(\ln n)^2} = 0$$

because $\ln n \to \infty$ as $n \to \infty$. The sequence converges, and its limit is 0.
b) $b_n = \sqrt[n]{6n^8}$
Answer $\lim\limits_{n\to\infty} \sqrt[n]{6n^8} = \lim\limits_{n\to\infty} \sqrt[n]{6}\left(\sqrt[n]{n}\right)^8 = 1 \cdot 1 = 1$. These are two of the limits you're supposed to know.

9. Determine whether each of the following infinite series converges or diverges. Explain your answers.
a) $\sum\limits_{n=1}^{\infty} \frac{1}{8n^6 - \frac{1}{n^2}}$
Answer This looks like a p-series with $p = 6$, and since $6 > 1$ I guess that the series will converge. More formally, I know that $\sum\limits_{n=1}^{\infty} \frac{1}{n^6}$ converges. I will use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\frac{1}{n^6}}{\frac{1}{8n^6-\frac{1}{n^2}}} = 8 \text{ (Just multiply top and bottom by } n^6.)$$

and observe that 8 is not zero. Then the series must converge. To use the Comparison Test itself rather than the Limit Comparison Test is more difficult because of the minus sign in the denominator. It can be done, but you've got to be somewhat careful: compare the original series to a suitable multiple of the appropriate $p-$series to take into account the term with the negative sign in the denominator.

b) $\sum\limits_{n=1}^{\infty} \frac{(3n)!}{(n!)^2(2n)!}$

Answer Use the Ratio Test. Since $a_n = \frac{(3n)!}{(n!)^2(2n)!}$, $a_{n+1} = \frac{(3(n+1))!}{((n+1)!)^2(2(n+1))!}$ and $\frac{a_{n+1}}{a_n} = \frac{\frac{(3(n+1))!}{((n+1)!)^2(2(n+1))!}}{\frac{(3n)!}{(n!)^2(2n)!}} = \frac{(3(n+1))!(n!)^2(2n)!}{((n+1)!)^2(2(n+1))!(3n)!} = \frac{(3n+3)!(n!)^2(2n)!}{((n+1)!)^2(2n+2)!(3n)!} = \frac{(3n+3)(3n+2)(3n+1)(3n)!(n!)^2(2n)!}{((n+1)n!)^2(2n+2)(2n+1)(2n)!(3n)!} = \frac{(3n+3)(3n+2)(3n+1)}{((n+1))^2(2n+2)(2n+1)}$. This has limit=0 as $n \to \infty$ since the degree of the top is 3 and the degree of the bottom is 4. Therefore the series converges.

10. Find power series centered at 0 (called the Taylor series or the Maclaurin series) for the following functions:
a) $\frac{1}{1+x^3}$ b) $\frac{5}{x-3}$ c) $\frac{\ln(1+x^2)}{x}$

Answer a) Use a geometric series with $a = 1$ and $r = -x^3$. The result is $1 - x^3 + x^6 + \ldots = \sum\limits_{n=0}^{\infty}(-1)^n x^{3n}$.

b) Since $\frac{5}{x-3} = \frac{-\frac{5}{3}}{1-\frac{x}{3}}$, we can use a geometric series with $a = -\frac{5}{3}$ and $r = \frac{x}{3}$. The result is $-\frac{5}{3} - \frac{5x}{3^2} - \frac{5x^2}{3^3} + \ldots = \sum\limits_{n=0}^{\infty} -\frac{5x^n}{3^{n+1}}$.

c) The derivative of $\ln(1+x^2)$ is $\frac{2x}{1+x^2}$. But $\frac{2x}{1+x^2}$ is the sum of a geometric series with $a = 2x$ and $r = -x^2$ since $\frac{2x}{1+x^2} = \frac{2x}{1-(-x^2)}$. Therefore $\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 + \ldots = \sum\limits_{n=0}^{\infty}(-1)^n 2x^{2n+1}$. We can integrate and get $\frac{2}{2}x^2 - \frac{2}{4}x^4 + \frac{2}{6}x^6 + \ldots = \sum\limits_{n=0}^{\infty}(-1)^n \frac{2}{2n+2}x^{2n+2}$. But is this equal to $\ln(1+x^2)$? The series and the function are both antiderivatives of the same series. But look: $x^3 + 17$ and $x^3 - 38$ both have the same derivatives and are not equal. In this case, check $\ln(1+x^2)$ and $\sum\limits_{n=0}^{\infty}(-1)^n \frac{2}{2n+2}x^{2n+2}$ when $x = 0$: both of them are 0 (I don't know any other way of doing this and this should be checked!). Therefore $\ln(1+x^2) = \frac{2}{2}x^2 - \frac{2}{4}x^4 + \frac{2}{6}x^6 + \ldots = \sum\limits_{n=0}^{\infty}(-1)^n \frac{2}{2n+2}x^{2n+2}$ and, finally, a series for $\frac{\ln(1+x^2)}{x}$ can be gotten by dividing by x: $\frac{\ln(1+x^2)}{x} = \frac{2}{2}x - \frac{2}{4}x^3 + \frac{2}{6}x^5 + \ldots = \sum\limits_{n=0}^{\infty}(-1)^n \frac{2}{2n+2}x^{2n+1}$.