Taylor's Theorem problems for 152:05–07 and 09–11 4/12/2007

These problems are almost all taken from previous exams in Math 152.
The wise student will learn how to solve *all of them.*

1. We want to replace $\sin x$ by $x - \dfrac{x^3}{3!}$ in some interval centered around 0 (that is, an interval like $-A \leq x \leq A$ with $A > 0$). Find an A so that the error committed doing such a replacement will be no more than $\pm .001$.

Note There is *no* unique correct answer to this question! You must show an error estimate verifying that your answer is correct.

2. a) Use the Taylor series for cosine to show that the function $\cos\left(\sqrt{x}\right)$ is equal to a power series centered at 0.

b) Use your answer to a) to show that $\displaystyle\int_0^1 \cos\left(\sqrt{x}\right)\,dx$ is equal to a certain alternating series.

c) Compute the fifth partial sum of the alternating series in b). Estimate the difference (the error) between this partial sum and the sum of the whole series. Compare the answer to the value of the integral obtained using the `fnInt` routine on your calculator.

3. a) Find the fourth degree Taylor polynomials centered at 0 for the functions $f(x) = 1 - \cos\left(3x^2\right)$ and $g(x) = \left(\sin(x^2)\right)^2$.

b) Use your answers to a) to compute $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)}$.

4. Use Taylor polynomials to find a specific polynomial $P(x)$ so that $\displaystyle\int_0^{1/2} x\cos(x^3)\,dx = \int_0^{1/2} P(x)\,dx \pm .001$. You are not asked to compute the approximating integral, but to explain why the polynomial you give provides the required accuracy.

5. Find the Maclaurin series for $f(x) = \displaystyle\int_0^x \arctan(t^5)\,dt$. What is $f^{(6)}(0)$?

6. Write (with summation notation) a finite sum (involving only integers and their products and quotients) which will approximate $e^{-.4}$ with an error less than $\pm 10^{-(1{,}000)}$.

Note You are <u>not</u> asked to compute the approximation. There is *no* unique correct answer to this question. You could write the approximation as $T_n(-.4)$, where $T_n(x)$ is the n^{th} Taylor polynomial for e^x centered at $a = 0$. Then estimate the error (with the n you will specify!) to show that $T_n(-.4)$ is actually within the desired tolerance of $e^{-.4}$.