Math 152 Review Problems for the Final Exam, Spring 2007

The final exam will be cumulative, and the problems below are only samples of some of the types of problems which may be asked on the final. You should also study the review sheets for the two midterms, as well as your lecture notes, homework problems, midterm exams and the workshop problems for the course.

1. Let C be the curve $y = x^4/4$, with $0 \leq x \leq 1/2$.

(a) Set up an integral for the length of C.

(b) Using the binomial series and term-by-term integration, express the integral in part (a) as a convergent infinite series. Give numerical values for the first three terms in the series and a formula for the general term of the series.

(c) Explain why the method of (b) wouldn't work to find the length of the same curve extending from $x = 0$ all the way to $x = 2$. Give an approximate value for this length, using the trapezoidal rule with $n = 4$ divisions.

(d) Given that $\left| \frac{d^2}{dx^2} \sqrt{1 + x^6} \right| \leq 13$ for all $0 \leq x \leq 2$, estimate the error in your approximation in (c).

2. The curve with parametric equations

$$x = 8 - 2t^2, \quad y = \sin \pi t, \qquad -4 \leq t \leq 4$$

crosses itself at the origin. Find the t values at which it crosses the origin. Find the equations of both tangent lines at the origin.

3. Sketch the cardioid $r = 1 - \sin\theta$. What percentage of its enclosed area lies above the x-axis?

4. Find the solution of the differential equation $\frac{dy}{dx} = y\left(\frac{x^2 - 4x - 9}{x^2 - 1}\right)$ with $y(2) = 3$. Give an explicit formula for y as a function of x. Graph the solution and determine the largest interval $A < x < B$ for which the solution exists.

5. Let R be the region in the *second* quadrant which is bounded by the curves $y = e^x$ and $y = 0$.

(a) Sketch the region R and find its area.

(b) Find the volume of the solids which result when the region R is revolved (1) about the x-axis; (2) about the y-axis. (Note that these integrals are improper.)

6. Evaluate the following indefinite integrals:

$$\text{(a)} \int x^3 \cos x \, dx \quad \text{(b)} \int \sqrt{25 - x^2} \, dx \quad \text{(c)} \int \sqrt{2x - x^2} \, dx \quad \text{(d)} \int \tan^{-1} x \, dx$$

7. Evaluate the following indefinite integrals:

$$\text{(a)} \int \frac{5x^2 + 4x + 9}{(x^2 + 4)(x + 1)} \, dx \quad \text{(b)} \int (\cos^3 x)(\sin^4 x) \, dx \quad \text{(c)} \int \tan^6 x \, dx \quad \text{(d)} \int \cos^4 x \, dx$$

8. Evaluate the following indefinite integrals:

$$\text{(a)} \int x^2 (\ln x)^2 \, dx \quad \text{(b)} \int (\tan^2 x)(\sec x) \, dx \quad \text{(c)} \int \frac{e^x}{1 + e^{2x}} \, dx \quad \text{(d)} \int e^x \cos(4x) \, dx$$

9. Use a geometric series to write the repeating decimal $5.373737\ldots$ as a quotient of two integers.

10. A bacterial culture contains 5 grams of germs at 1:00 pm. It contains 12 grams of germs at 4:00 pm. How many grams of germs does it contain at 3:00 pm? Assume that the rate of growth of the germs at any time is proportional to the amount of germs present at that time. Give exact answers, not decimal approximations.

11. Assume that N is a natural number greater than 2.

(a) Show that $\displaystyle\sum_{n=N+1}^{\infty} \frac{\ln n}{n^2} < \int_N^{\infty} \frac{\ln x}{x^2} \, dx$ by drawing areas related to the graph of $y = \dfrac{\ln x}{x^2}$.

(b) Prove that $\displaystyle\sum_{n=N+1}^{\infty} \frac{\ln n}{n^2}$ converges.

12. Suppose you need numerical values of a function $f(x)$ defined by a very complicated formula. You know, however, that $f(3) = 1$, $f'(3) = -2$ and $f''(3) = 20$. Moreover you know that the third derivative of $f(x)$ satisfies $|f'''(x)| \leq 24$ for all x in the interval $2 \leq x \leq 4$. Compute the second-degree Taylor polynomial T_2 for f centered at 3. Use it and Taylor's Inequality to solve the following problems.

(a) Calculate the best approximate value for $f(3.3)$ that you can from this information, and then estimate the error.

(b) Find a number $B > 0$ so that $|f(x) - T_2(x)| \leq 1/10$ for *all* numbers x in the interval $3 - B \leq x \leq 3 + B$.

13. Let $f(x) = \cos(3x)$ and $g(x) = e^{x/2}$.
(a) Find the coefficients a_0, a_1, a_2 in the Maclaurin series $f(x)g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$.
(b) Find the coefficients b_0, b_1, b_2 in the Maclaurin series $\dfrac{f(x)}{g(x)} = b_0 + b_1 x + b_2 x^2 + \cdots$.

14. Evaluate $\displaystyle\lim_{x\to 0} \frac{(e^x - 1 - x)^3}{(2x - \sin(2x))^2}$. Do not use l'Hôpital's Rule.

15. Find the Maclaurin series for the function $f(x) = \dfrac{x^7}{8 + x^3}$. Use the result to find an infinite series representation for $\displaystyle\int_0^1 f(t)\,dt$. Estimate the size of the difference between this integral and the 3rd partial sum of the series.

16. For each series below, determine whether it is absolutely convergent, conditionally convergent, or divergent. In each case give details to support your answer and indicate which convergence test you are using.

(a) $\displaystyle\sum_{n=8}^{\infty} \frac{(-1)^{n+1}}{\ln n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{\tan^{-1}(n^3)}{n^{5/4}}$ (c) $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)^n}{n^{1/n}}$

17. For each series below, determine whether it is convergent or divergent. In each case give details to support your answer and indicate which convergence test you are using.

$$\sum_{n=3}^{\infty} \frac{7^n + n^2}{8^n + n}, \quad \sum_{n=5}^{\infty} \frac{\cos(n^2 + e^n)}{\sqrt{n^3 - n - 1}}, \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

18. Use comparisons to determine whether the following improper integrals are convergent or divergent.

$$\text{(a)} \int_0^{\infty} e^{-x^4}\,dx \qquad \text{(b)} \int_0^{\infty} \frac{x}{x^8 + x^5 + 2}\,dx \qquad \text{(c)} \int_0^1 \frac{2 + \sin(e^x)}{x}\,dx\,.$$

19. Show that $\displaystyle\sum_{n=1}^{\infty} \left(\sin^2\left(\frac{\pi}{n}\right) + \cos^2\left(\frac{\pi}{n+1}\right) - 1 \right) = 0.$

20. Determine the radius and interval of convergence of each of the following power series. In addition, determine those points at which each series is absolutely convergent.

$$\text{(a)} \sum_{n=2}^{\infty} \frac{2^n (x+3)^n}{\sqrt{n}}, \qquad \text{(b)} \sum_{n=3}^{\infty} \frac{(-1)^n (x-1)^n}{n^2 + 1}\,.$$

21. Find the Maclaurin series of $\sin^{-1} x$ using $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$ and a binomial series.